Captain Astounding's

NIGHTCLUB

THE POWER

A STORY ABOUT WHAT HAPPENS WHEN YOU START MULTIPLYING NUMBERS BY THEMSELVES...

...IN A NIGHTCLUB

Captain Astounding's Nightclub is a series of books about mathematics, science, and technology. Far from being text books, they concentrate on the real meaning of the concepts and on their practical uses. Students of all kinds will enjoy them, especially mature students brushing up on long forgotten school topics or those venturing into the more numerate subjects such as information technology.
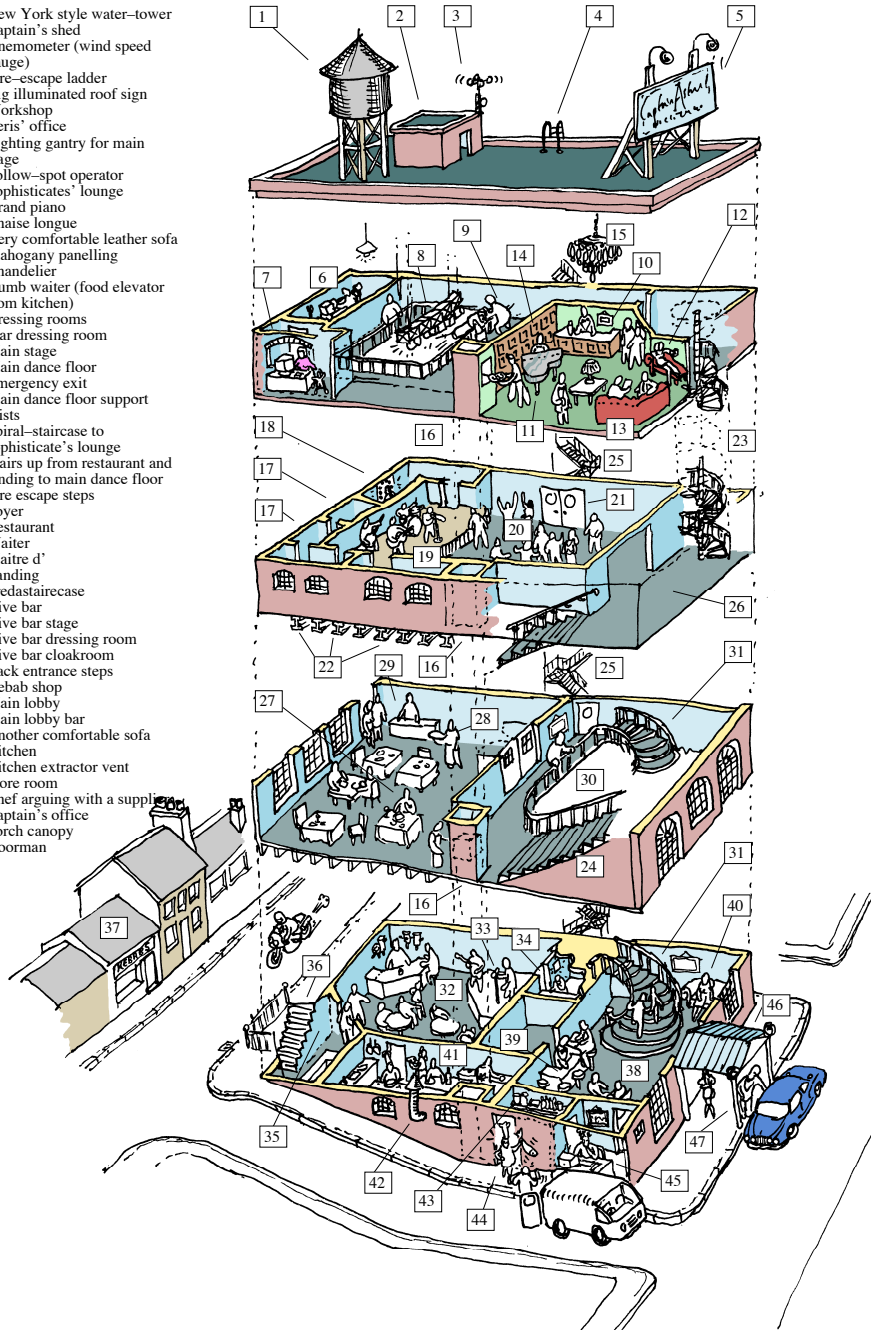
DAN WELCHMAN
PRODUCTIONS

The Power

Diagram A: Exploded view of Captain Astounding's Nightclub

1. New York style water–tower
2. Captain's shed
3. Anemometer (wind speed gauge)
4. Fire–escape ladder
5. Big illuminated roof sign
6. Workshop
7. Ceris' office
8. Lighting gantry for main stage
9. Follow–spot operator
10. Sophisticates' lounge
11. Grand piano
12. Chaise longue
13. Very comfortable leather sofa
14. Mahogany panelling
15. Chandelier
16. Dumb waiter (food elevator from kitchen)
17. Dressing rooms
18. Star dressing room
19. Main stage
20. Main dance floor
21. Emergency exit
22. Main dance floor support joists
23. Spiral–staircase to sophisticate's lounge
24. Stairs up from restaurant and landing to main dance floor
25. Fire escape steps
26. Foyer
27. Restaurant
28. Waiter
29. Maitre d'
30. Landing
31. Fredastairecase
32. Dive bar
33. Dive bar stage
34. Dive bar dressing room
35. Dive bar cloakroom
36. Back entrance steps
37. Kebab shop
38. Main lobby
39. Main lobby bar
40. Another comfortable sofa
41. Kitchen
42. Kitchen extractor vent
43. Store room
44. Chef arguing with a supplier
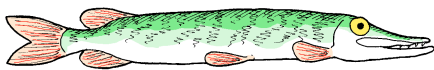45. Captain's office
46. Porch canopy
47. Doorman

Captain Astounding's Nightclub

THE POWER

AN EVERYDAY STORY
ABOUT WHAT HAPPENS
WHEN YOU START
MULTIPLYING
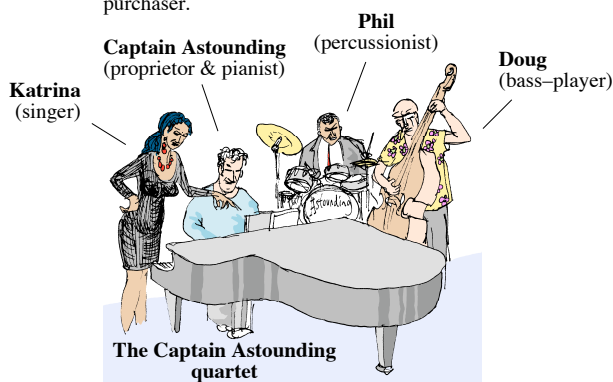NUMBERS BY
THEMSELVES

...IN A NIGHTCLUB...

DAN WELCHMAN
PRODUCTIONS

Katrina
(singer)

Captain Astounding
(proprietor & pianist)

Phil
(percussionist)

Doug
(bass–player)

**The Captain Astounding
quartet**

Chef
(chef)

for Paul Freedland

The various differently coloured sections in this book (and others in the Captain Astounding series) are used to indicate interesting peripheral topics outside the main flow of the text.

Terminology

A section in a yellow box like this will be about what things are **called** rather than what they mean.

In technology (and especially in Mathematics) one single concept often goes by many different names. Some people will use one term while others use another. To make matters worse, they also sometimes use the same word to mean two completely different things. So you'll be seeing quite a few of these yellow sections. They should help to avoid confusion.

On your calculator

These words can appear in any of the coloured boxes (or even outside them). When you see them you'll be about to learn how to actually work something out on your calculator. This happens quite frequently, as most of the concepts in this book are directly applicable to real life problems. A calculator application on your computer will serve just as well.

Sophisticates' lounge

This elegant shade of green is the colour of the sophisticates' lounge. That's the rather refined establishment situated on the top floor of the nightclub. The talk in here is all super-advanced stuff of a most sophisticated nature. It's daring and extravagant material that you're not expected to understand completely at first (if at all!) but it's just so juicy that we couldn't bear to leave it out.
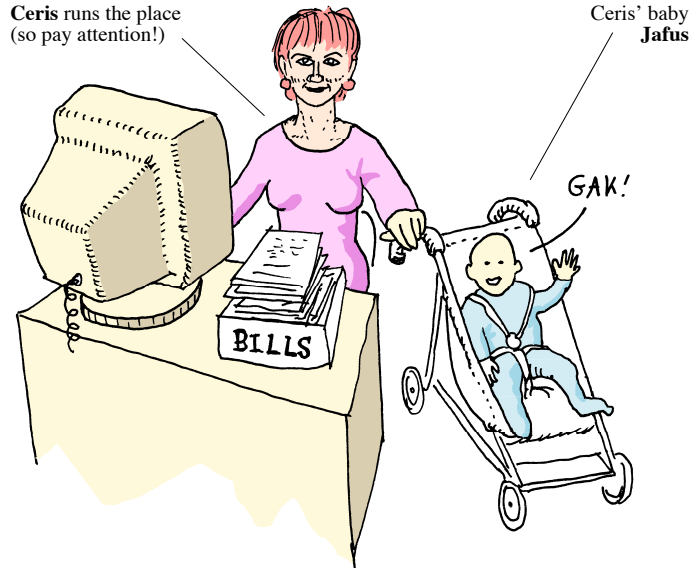
You don't **have** to go here (in other words you can skip these sections) ...but you are cordially invited.

Oh **do** hurry up dahling!
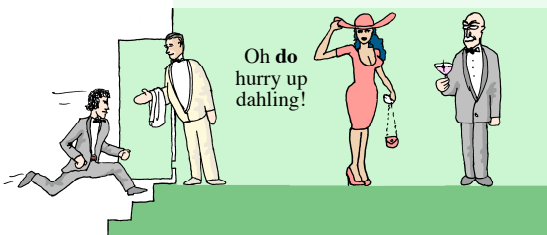
The band don't come up here very often but when they do they tend to get a bit pompous and start calling each other "dahling", don't ask me why.

How to use this book

Ceris runs the place (so pay attention!)

Ceris' baby **Jafus**

GAK!

BILLS

Exercises

Pink means exercises. We put these in for the benefit of people who would like to test out whether they've really understood what Captain Astounding and his colleagues have been talking about. You can call them examples or problems or tests or whatever you like. They're not obligatory, just offered intermittently to allow you to flex your muscles, should you feel the need.

If you're already bursting to have a crack at an exercise then why not try to solve this one:- Where are the answers to the exercises found, and what do they look like?

Background

Pale blue sections like this one are for background reading. This is where you'll find interesting bits and pieces that aren't part of the main flow of the story. Life *does* exist outside this nightclub after all! (even if the Captain doesn't believe it). You can skip over the blue bits if you like, but you might miss out on some amusing anecdotes and fascinating facts.

A bit of history

This deeper blue is reserved for **Historical** background. It'll appear whenever there's an explanation of how a particular concept was discovered or invented. It'll also crop up for interesting applications used in the past but no longer in use today.

On the moon?

If you thought that the answers were on the moon then you probably need medical help.

...but you've given me the chance to demonstrate that we often give some common wrong answers as well, with an explanation why they're wrong.
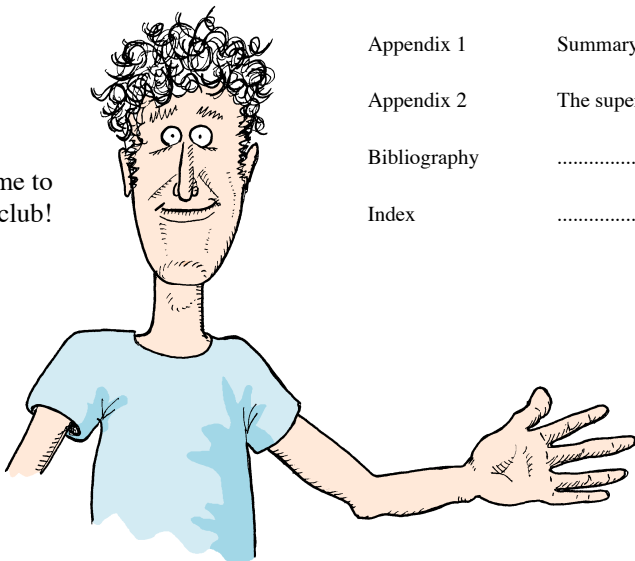
At the bottom upside down?

If you thought that the answers were here at the bottom upside-down in a tiny font then you were right.

Congratulations

Contents

Welcome to my nightclub!

Acknowledgements

Expensive paint and Jungle Juice

The "square" and "cube" of a number

This tin of incredibly expensive iridescent paint is part of my latest cunning plan to liven up the main dancefloor. I can't afford to paint the whole floor ^{see note i} so I'm just going to mark out a diamond in the middle. In fact, the diamond is just a big square turned round diagonally a bit. Now it says on the tin that the paint will cover 20 square metres, so we ought to be able to get a nice big diamond out of it. If I paint a diamond that's four metres wide then the area I'll need to cover with paint will be four times four square metres (or four "squared") so that's sixteen square metres in total. So there should be plenty of paint to do that with a little bit left over. If I'd chosen a three metre wide diamond then that would only have 3^2 (= 3 X 3 = 9) square metres so I'd have had enough paint for **two** diamonds if I wanted! But I prefer the look of a single big diamond.
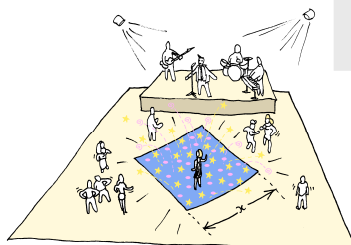
COVERAGE **20** SQUARE METRES

What about a bigger one?

Now here's a puzzle for you. Even with the big 4 metre diamond I get a bit of paint left over, so I'm not really getting my money's worth[see note ii]. Is there enough for a 4½ metre diamond?

$$4^2 \quad = \quad 4 \times 4 \quad = \quad 16$$
(4 squared)

4   4

Did you say "No! there's not enough paint!"?

…that's the right answer. A 4½ metre diamond will take 4.5 x 4.5 (that's 4.5 squared) square metres worth of paint. That's 20.25 square metres in all, so it's a close call but there's not quite enough paint to do it

Did you say "Yes! there IS enough paint to do it!"?
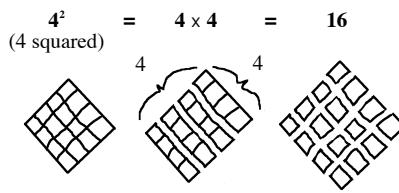
…wrong answer. I'm afraid. The 4½ metre diamond covers 20¼ square metres (see correct answer opposite) so you can't quite do it with one tin. Maybe you calculated that since the square was 0.5 metres bigger (in other words ⅛th as big again or 12.5% larger) then you'd need that much more paint. That would be 18 square metres (4 X 4.5) so you'd be laughing. But you've got to make **both** sides of the diamond that much bigger –that's where the extra paint goes!

Terminology

Well as you've probably guessed by now (if you didn't know already) the result of multiplying a number by **itself** often gets called the "square" of that number, or the number "squared". It doesn't matter if it's a neat whole number (like 4) or some other number (like 3.456) it still just means that number times itself.

In the case of my lovely paint diamond you can easily see why the word "square" gets used because it's a physical square on the ground that we're talking about. But there are lots of other times when we need the square of a number that have nothing to do with shapes or areas

When you see it written down, x squared often appears in a sort of shorthand where the number has a little digit 2 just beyond and above it like this:- x^2

The little 2 is there to tell us that the x appears **twice** in the full length version, $x \times x$.

On your calculator

On your calculator you can work out x squared by simply typing the number followed by the x key followed by the number again and then = (just like you would multiply any two numbers). But unless it's very basic you'll probably have an x^2 key that does it with one press!

Well apparently my beautiful paint diamond is considered a bit boring [see note i] by some people. Ceris and Phil have come up with a couple of alternative designs that are a lot more complicated and fussy. They're still based on the (brilliant) idea of a diamond shaped patch of paint, but they've got certain parts cut out of them. Both designs divide the diamond up into 9 squares and then rearrange them for a decorative effect. That's not to say that we'll necessarily paint them on a 3 X 3 square metre plan (in fact, we haven't decided the exact size yet), that's just the way that Ceris and Phil have chosen to break up their diamonds. We'll just call the size x until we finally make up our minds what it will be. (Numbers that you haven't yet fixed a value for always get called x, don't ask me why).
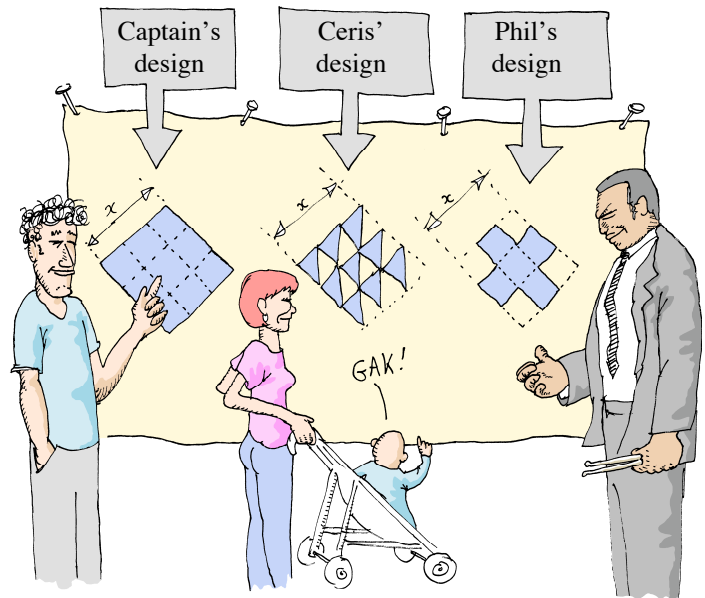
Ceris' design works by taking each of the smaller squares and chopping it in half diagonally. Then she discards half of them leaving a flock of nine little triangles.

…it reminds me of a fish!   or, maybe the "Red Arrows"!
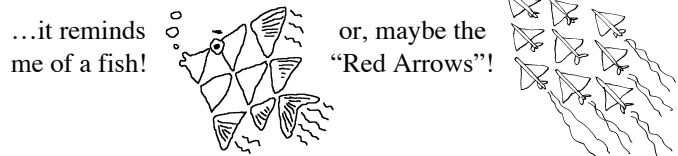
In Phil's design the four corner squares are cut away leaving the remaining five on their own. The resulting shape is a kind of diagonal cross.

Note i: I thought it was quite good, actually! —*Captain*

Just because these designs divide the diamond up into three rows and three columns doesn't mean that I've got to paint them three metres wide. For example, we could paint them four metres wide (see diagram right) so that each individual small square would be 1⅓ metres in width. What I'm really interested in is an answer that will tell me how much paint I'll need no matter what size (x) I choose.

Question   How much paint do Ceris and Phil need for their designs?

So here's a question (or two) for you. Given that my original diamond design covers x^2 square metres when it's x metres wide, what area will need to be covered in paint for **a**) Ceris's design and **b**) Phil's design?

I don't want the answer for some specific size like 4m or 5m, I'm looking for the **general** answer in terms of the overall width x.
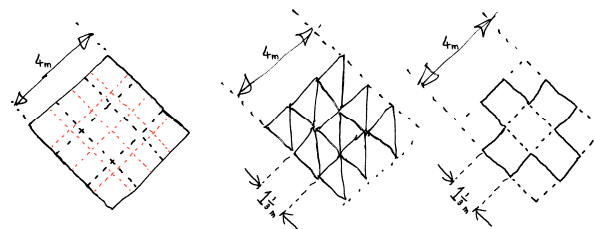
Well that's enough about expensive paint and squaring numbers for now. We'll come back to it later on in chapter 6 when we look at working out a diamond that's perfectly sized to use up all of the paint in the tin with none left over[see note i]. The important thing is that when we increase the size of the diamond the area covered (and so the amount of paint) goes up by that increase squared. Ceris may have come up with a design that's more economical on paint, but if we double the size of the diamond it still needs four times as much paint as her original. Even if we only make the diamond 10% bigger (in other words multiply it by 1.10) the paint needed goes up by a factor of 1.1 X 1.1 which is 1.21, so we need more than 20% extra paint to do it! It works the same way with Phil's design of course– it uses a little bit more paint than Ceris's and a little bit less than mine, but when he makes it 10% bigger he still needs 21% extra paint over whatever the original amount was.

a) Did you get $1/2x^2$ square metres for Ceris's design?

You got the correct answer if you did! Each of those triangles in Ceris's design is a square chopped in half so it has half the area of the same part of my design. So it doesn't matter how big the diamond is:– Ceris's design always uses half as much paint as mine.

b) Did you get $5/9x^2$ for Phil's design?

Well that's the right answer (although the decimal version 0.5555556x would have done almost as well). Phil cuts away four of the 3 X 3 squares that make up the complete diamond. This leaves five behind making up his diagonal cross. So the cross only covers five–ninths as much area as my plain diamond. So he uses more paint than Ceris but less than me, if we all lay out our designs at the same width x.

Note i: How many times have I got to tell you! the man's a cheapskate!

Mind you, this stuff does look rather nice on my nails!

Katrina

Terminology

When you multiply a number by itself and then by itself again the result is usually called that number "cubed". You can see where the name comes from when it's to do with working out the volume of something (like our juice cartons). When you make an object that has the same width and depth and height then you naturally end up with a cube shape. But cubed numbers are useful for working things out about lots of other shapes too. They're even used for things that have nothing to do with physical size and volume (see the bike example in the next chapter).

When you see it written down a number cubed is often written as the number followed by a little figure 3 just after and above it like this:- x^3

It's similar to x^2 –the little 3 tells us that the number appears 3 times in the longhand version:
$x \times x \times x$

Jungle Juice   and the CUBE of a number

Behold the latest ingenious money–making scheme of a business mastermind (if I do say so myself)! Chef has been making his delicious home made Jungle Juice cocktail for years now. We sell bucket loads of it in the dive bar and even in the sophisticate's lounge. But now we're going to make it available in these attractive cartons (just like the ones you get milk in) so the customers can take some away to drink at home!

The cartons come in all sizes but they're always the same shape. The height is always twice the width of the square base. So the shape is like two cubes stacked on top of each other (ignoring the funny triangular bit at the top where you open it). The question is; how much Jungle Juice will you get in each carton? Well let's consider just one of the two cubes for a moment. If it's three centimetres wide then it'll be three centimetres deep and three centimetres tall as well. So if you imagine all the little 1cm cubes that fill up the shape you can see that there will be three layers of 3 X 3 little cubes of Jungle Juice. That gives a total volume of 3 X 3 X 3 (= 27) cubic centimetres in all. But we might not want a 3cm wide carton (which is on the skinny side after all). We might want it 4cm wide (giving 4^3 = 4 X 4 X 4 = 64cc) or even some arbitrary size x centimetres wide, so the cube will contain $x \times x \times x$ or x^3 cc of juice.
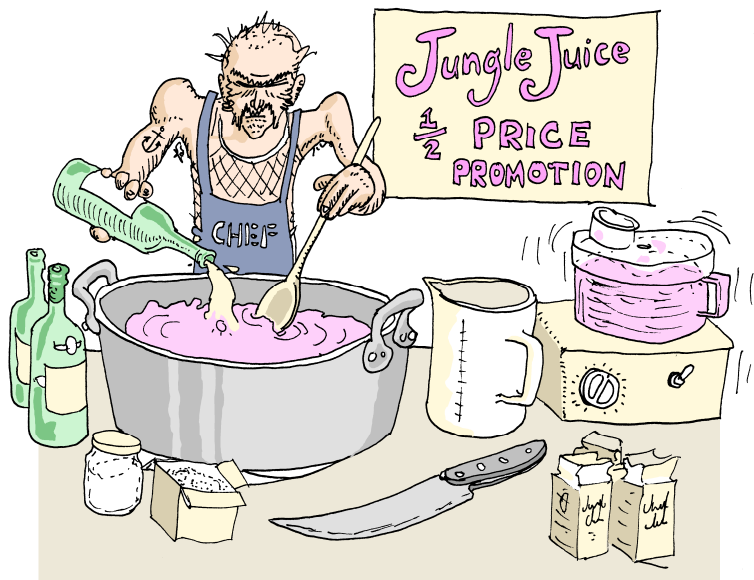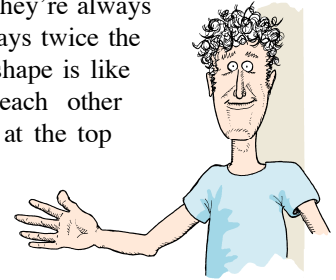
The height of our Jungle Juice cartons is double the width though, so it's like having **two** of those 3 X 3 X 3 cubes stacked on top of each other. So the total amount of juice will be 2 X 27 = 54 cubic centimetres (or 54 millilitres, which is the same thing)[see note i]
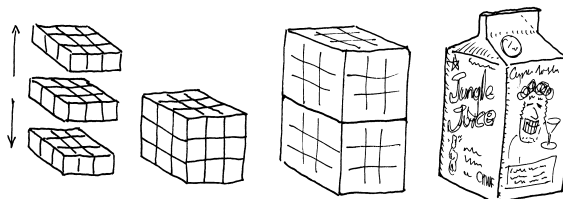
How much juice do you get?

Here's a question for you then: How much cocktail do you get in a carton? I'm not talking about the 3cm carton specifically –I'm interested in how much you get generally in one of any size *x*.

Ah! but what about the juice in the triangular part at the top?

Ceris

Ha! I knew you'd say that! –but Chef doesn't fill the cartons right up to the top so that the triangular bit is full. He only pours juice in up to the level of the rectangular section. If he didn't the juice would splurt out in your face when you opened it.

(in fact they sometimes do anyway!)

I thought you were the sensible one!?

Well 54ml is only about half a wine glass full so that's not much use! (I wouldn't want to get called a cheapskate!). If we bump the size up a bit to 5 centimetres then we'll get 2×5^3 ml which is equal to 2 X (5 X 5 X 5) which is 250ml. That's much better!

On your calculator

Most decent calculators have an x^2 function (except for the ones on mobile phones — which is a bit disappointing) but it's much less common to have an x^3 button unless its one of those calculators with loads and loads of buttons. So you may be stuck with typing in "*x* X *x* X *x* =" —longhand.

There **is** one other possibility though, and that's if the calculator has a y^x button. But that's really one for the sophisticate's lounge —we'll return to it later, in chapter 3.

Did you get $2x^3$?

Right again! Fantastic! It's the volume of two cubes which are *x* wide. So yes, the answer is $2x^3$

Did you get $3x$?

That's not the right answer. You've multiplied *x* by the number 3. What we need to do is multiply *x* by **itself** (and then again) so it's *x* times *x* times *x*.

Did you get x^3 ?

Not quite right! –you were right to put an *x* cubed in there. But because the carton is twice as tall you need **two** of them to get the right volume.

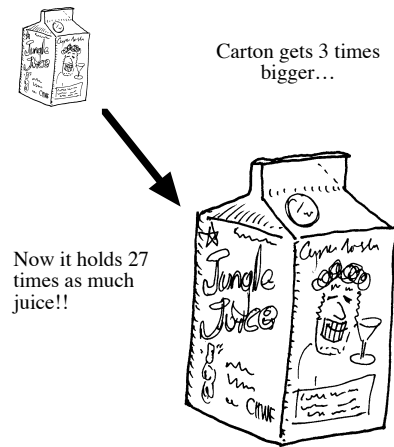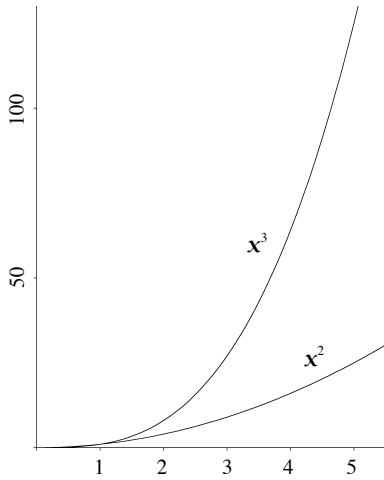**Note i**: ml stands for millilitres and cc stands for cubic centimetres (and yes, they're exactly the same volume)

x^3

x^2

Carton gets 3 times bigger…

Now it holds 27 times as much juice!!

We know from messing about with the iridescent paint that the square of a number goes up pretty sharply as the number gets bigger. When you double the number the square goes up **four** fold; if you triple the number it goes up **nine** fold; so x squared rises in a curve as x increases.

But x cubed rises in a much steeper curve even than that! It's amazing to think that if we take our standard jungle–juice carton (about two wine glasses full) and just treble the size of it from 5 to 15 centimetres you end up with twenty–seven times the volume. That's 6750 cubic centimetres or about 54 wine glasses full —enough for an entire party!!

If we look at something that only goes up with x squared (like the amount of paper needed to make one of these jumbo party cartons) then that will only be nine times as much.

And there's another thing that I find quite astounding; even though we don't know exactly how much paper was needed to make the standard 5cm carton (due to fiddly bits like the triangular top) we still know that the 15cm carton is going to need nine times more! With the volume of juice we happened to know exactly how much was there (it was 250cc) but imagine if we'd had a strangely shaped carton where we didn't know the volume, for example a ball–shaped one[see notes i & ii]. If we sell a 15cm party ball that's three times the diameter of the standard 5cm ball then it's **still** going to contain twenty-seven times as much delicious juice! This principle applies no matter what the shape of the carton you choose –just as long as you don't change the shape when you make the jumbo sized one.

Note i: Mind you, it's a tad tricky making a spherical carton out of paper

Note ii: …and they have an annoying tendency to roll off the table!

Bendy joists and hurricanes

Applications which are not to do with areas, volumes or shapes. Higher powers beyond x^3

Hmmm...
...getting a bit windy!
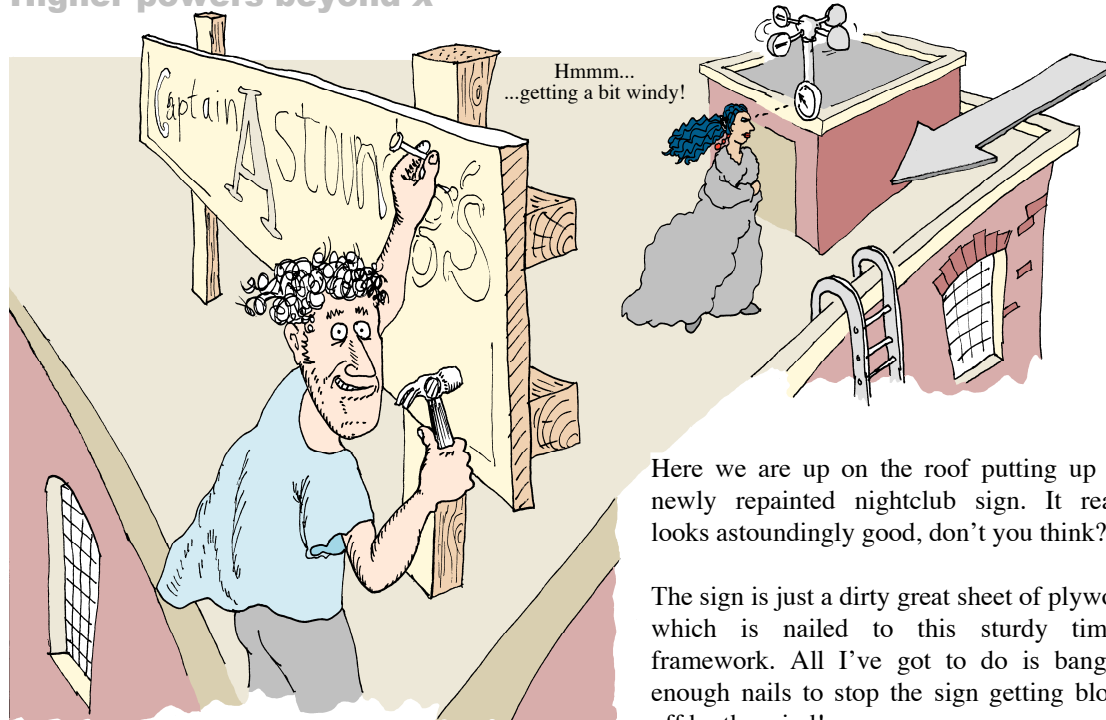
Here we are up on the roof putting up my newly repainted nightclub sign. It really looks astoundingly good, don't you think?

The sign is just a dirty great sheet of plywood which is nailed to this sturdy timber framework. All I've got to do is bang in enough nails to stop the sign getting blown off by the wind!

Now obviously the faster the wind blows the harder it pushes on the sign. Everyone knows that. But not so many people know that the hardness of push goes up as the **square** of the wind speed, oh yes. So if the wind suddenly gets twice as fast then the force on the sign isn't just doubled; it increases by a factor of four!
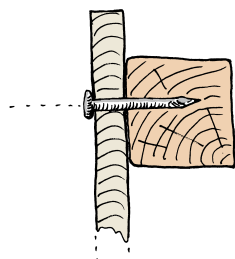
And if the wind speed triples then the force goes up by a factor of 9 (because $3^2 = 9$) so I'd suddenly need nine times as many nails to stop the sign blowing away!

Wind pressure is just one of many phenomena where x^2 is useful for working out what's going to happen, it's not just for shapes and areas!
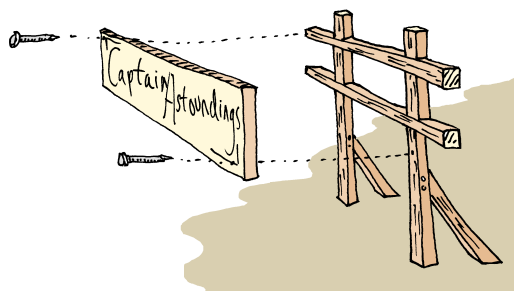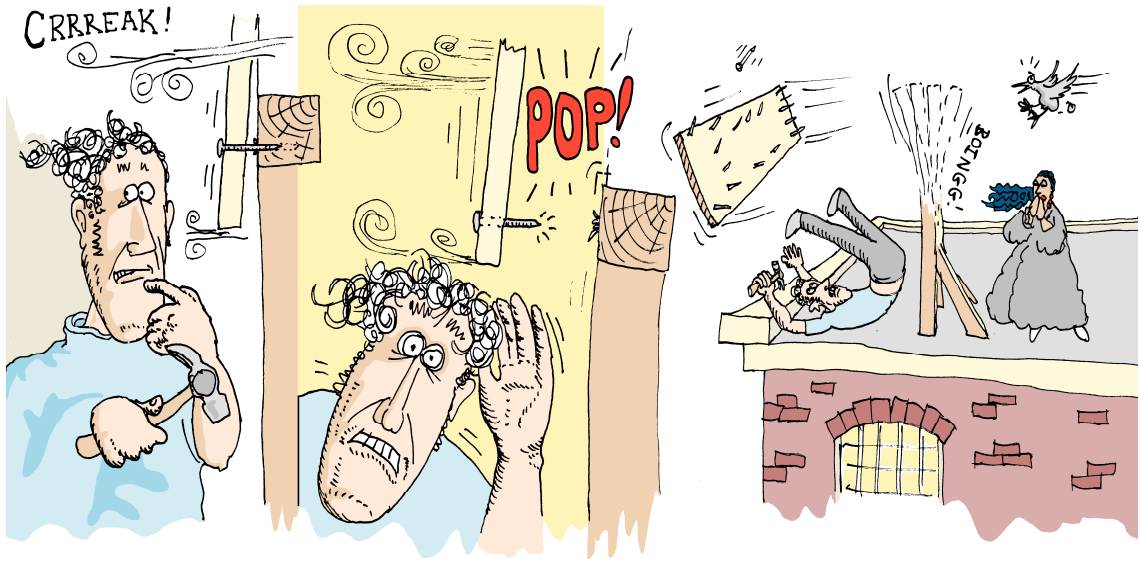
An exercise

As soon as the wind exceeded 40 miles per hour the sign blew off. That was with 20 nails holding it on. In other words 20 nails were **just** adequate to hold it on at 40mph. how many nails would be needed to **just** keep the sign on in a 100mph hurricane? (see diagram A below). You can assume that each nail is identical and taking exactly the same share of the overall wind force.

Gahhhh!! —maybe I should have waited for a less windy day before trying to nail up the new sign!

Katrina was keeping an eye on the anemometer[see note i] and she says that it was a gust of wind that just exceeded 40 miles per hour[see note ii] that managed to blow the sign off. Luckily I ducked out of the way just in time[see note iii]. I'd just hammered in the 20th nail and they were spread out pretty evenly over the framework too. So I reckon each nail was doing it's share and taking one twentieth of the wind load. The problem was that there just weren't enough nails to hold the sign on in a wind faster than 40mph. If it'd only been a 30mph wind then twenty nails would have been more than enough.

Diagram A
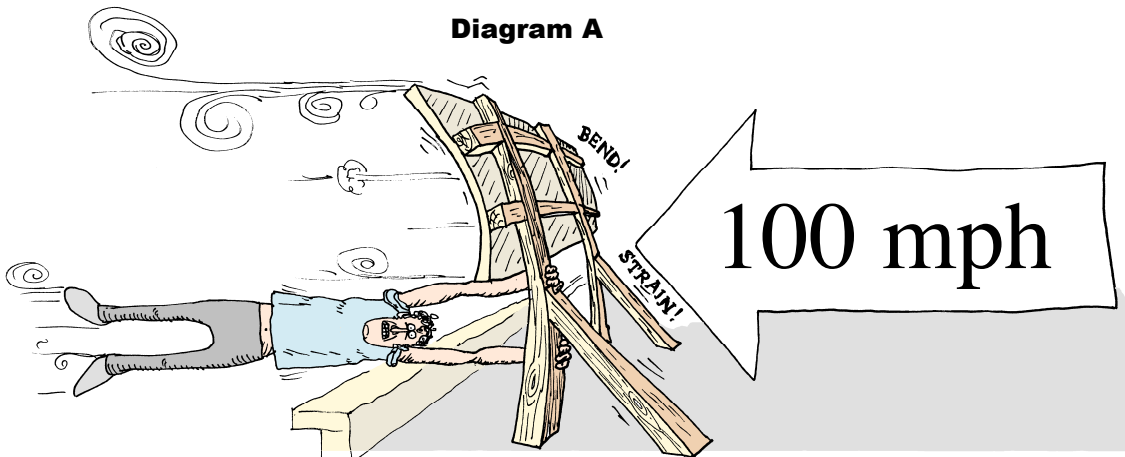
Did you get 80 ?

Maybe you multiplied my 20 nails by a factor of $2^2 = 4$. But the hurricane is 2.5 times faster than 40mph. So you should have multiplied it by $(2.5)^2 = 6.25$. You need more nails than that!

Did you get 50 ?

Well that's 2.5 times as many nails as I used, for a wind that's 2.5 times faster. But the wind pressure grows as the **square** of wind speed— so it's wrong!

Did you get 125 ?

That's right. 100mph is 2.5 times faster than 40mph so the force is $(2.5)^2 = 6.25$ times greater. So we need $6.25 \times 20 = 125$ nails!

Note i: That's the wind speed meter on top of my shed up here on the roof

Note ii: It's an old pre–metric anemometer hence the imperial units. Give us a break!

Note iii: No nightclub owners were harmed or maltreated during the making of this chapter

A tale of two motorcycles
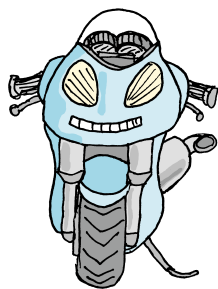
Have you ever wondered why 1000cc motorbikes **don't** go twice as fast as 500cc bikes? (They go faster for sure, but the top speed is nowhere near double). Well in real life the exact answer is pretty difficult due to lots of interesting little complications. For one thing the 1000cc bike probably wouldn't be the same size and shape as the 500cc one. And for various reasons the 1000cc engine might not give quite as much as double the power of the 500cc one. But even if the 1000cc bike was exactly the same size and shape and was exactly twice as powerful then it **still** wouldn't go twice as fast.
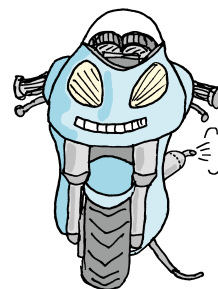
Take these two bikes for instance. They're both very nearly the same size and shape, in fact from the front you can hardly tell them apart. But the bike on the left is Doug's 1000cc super Bass Burner special of 120 horsepower[see note i] whereas the other one is my not–very–astounding 500cc Nightclub Ninja which is exactly half as powerful at 60 horsepower. Looking at them from the side you can see the difference.

 Now most of a motorcycle engine's power (not all, but most) goes into simply pushing the bulk of the bike though the air, especially when going fast. So it's not so different from the nightclub sign in the wind. The sign stays still while the wind rushes past whereas the bike rushes through the still air. but in both cases the wind pressure goes up with the square of speed so doubling the speed results in four times the force.

Doug's
120hp
1000cc
bike

My
60hp
500cc
bike
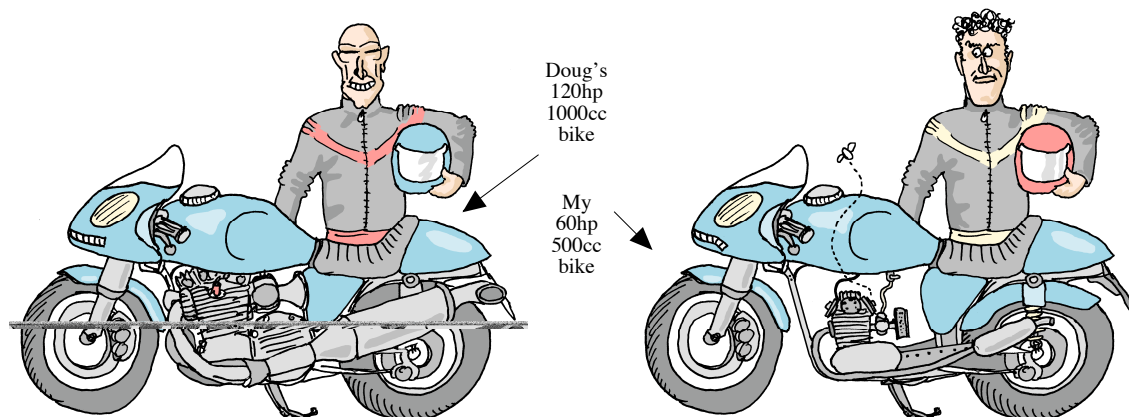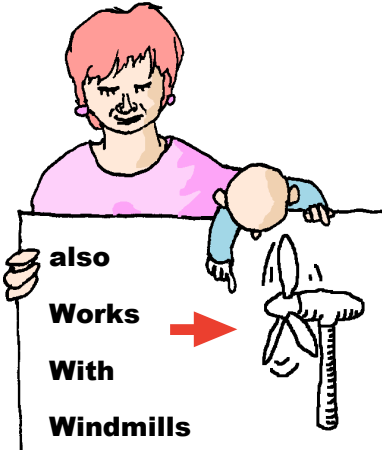
This means that the drag force holding you back goes up really sharply as you go faster. Anyone who has ridden a fast bike will know that this is true because you can feel your head being pushed backwards by the resistance of the air rushing past your helmet. And it gets rapidly worse as your speed rises. It's literally a pain in the neck, and it's no coincidence that most really quick bikes have fairings that you can crouch behind to get out of the air blast.

But if you look at the power required from the engine as the speed rises you will find that the situation is even more extreme!

Note i: A quick note about the word "horsepower": it's just an old fashioned unit for describing how powerful an engine is. These days we should really be using Watts (W) or Kilowatts (kW). One horsepower is roughly 750 Watts (or three–quarters of a Kilowatt, if you prefer). But Kilowatts and horsepower are both just measures of how much energy you get per second. It's a completely different meaning of the word "power" from the business of multiplying numbers by themselves.

also

Works

With

Windmills

Spookily enough, the amount of power that you can get from a wind turbine also goes up with the cube of wind speed. It's for similar reasons too. The force on the blades due to wind pressure goes up with the square of wind speed, but the sheer volume of air going through the blades increases in proportion at the same time. So you get x^3 times the power when the wind speed rises by x. This makes designing wind power systems a bit tricky!

We know that a bike doing 120mph (two miles per minute) will experience **quadruple** the drag force it gets at 60mph (one mile per minute). But it also goes twice as far in the same time. So the poor old engine has to push the bike double the distance and four times as hard. This means that the actual horsepower needed (and the amount of fuel you get through) goes up by a factor of **eight.**

drag

60 mph

1 mile

4 × drag

120 mph

2 miles
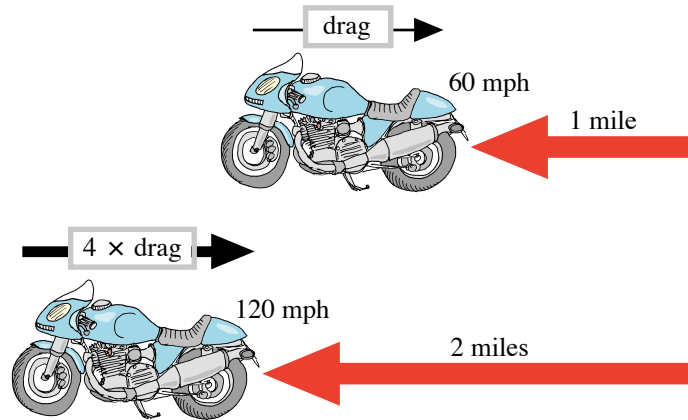
In fact if we increase the speed by any amount x the horsepower needed always goes up by a factor of x^2 (for the increase in wind drag) multiplied by x again (for the extra distance). So it's a factor of x^3 you need to multiply by to find out the extra engine power required to get a certain increase in speed.

No wonder 1000cc bikes don't go twice as fast as their 500cc little brothers!

The 220mph bike

Well my old 500 will just about manage 110mph flat out on it's 60 horsepower, but as we know that doesn't mean Doug's 1000 will be capable of 220mph just because it can can churn out 120hp. So here's an exercise for you: If we did want a 220mph bike[see note i] how powerful (in horsepower) would the engine have to be?

Did you get 240hp ?

That's what you'd get by **squaring** the increase in speed. The speed has doubled so $x^2 = 4$ and four times 60hp is 240hp. But the engine power needed goes up with the **cube** of speed, so we need to multiply our 60hp by $2^3 = 8$.

Did you get 360hp ?

Maybe you confused x^2 with x × 3.

…but they're not the same thing. If you multiply the speed increase (2) by three you get (60hp times 6 = 360hp) but the right answer is 60hp times 2 cubed (which is 60hp times 8 = 480hp)

Did you get 480hp ?

Bang on. The speed went up by a factor of 2 (from 110 to 220mph) so the engine power had to go up by a factor of 2^3 (= 8) to 480hp

Note i:…if we were bonkers enough to actually want one.

The infamous bouncy dancefloor

More problems, would you believe it?! —this is Ceris and myself underneath the main dancefloor and just look at the way it's bouncing around and sagging in the middle! Now a dancefloor is supposed to have a bit of "give" but this is excessive (at least that's what Ceris says and you don't argue with Ceris even **before** she has read the building regulations).
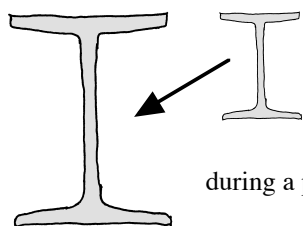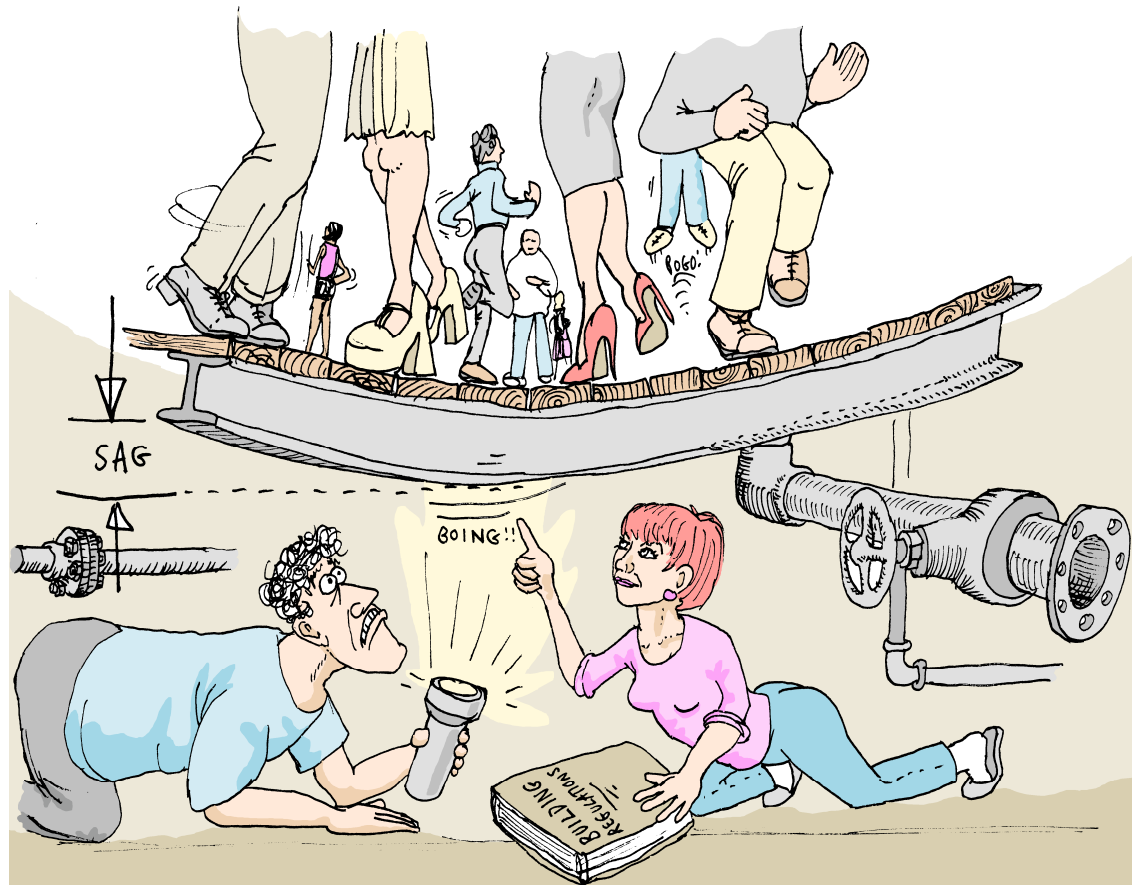
The problem isn't that the beams aren't strong enough (they're not actually breaking) but that they're not stiff enough. This means that they simply bend too much for a given weight of dancers above.

We just took a measurement of the maximum sag[see notes i and ii] at mid bounce during a particularly popular fast number and it was 200mm —way too much!

So it looks like I might be faced with the very expensive option of putting some new stiffer beams in to keep the maximum deflection within allowable limits. Now not many people know this, but as you scale up the cross section of a beam by a certain multiple x its stiffness goes goes up by a factor of $x \times x \times x \times x$ which is often written (surprise, surprise) as x^4 and called "x to the power 4" or "x to the 4th power" (or even just "x to the 4", for short).

To buy or not to buy?...   ...that is the question.

Well the building regs say that a dancefloor shouldn't bounce by more than 20mm (that's 2cm to you and me). Now it just happens that Phil's mate Derek from down the pub has a load of beams going cheap that are exactly twice the size of ours (in both width and depth). If we buy Derek's beams and install them will our problems be over?   *(Answers on the next page)*

Note i: "Sag" is what engineers call a downward deflection

Note ii: not to be confused with Sag Alloo which is a delicious Indian dish that Chef sometimes makes
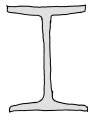
Here's our original dancefloor **I**–beam

...and here it is doubled in size making it **sixteen** times stiffer

...this one's doubled in height but we've kept the width the same. It's still **eight** times stiffer than the original

(note that doubling the height has made the flanges thicker too)

...this one hasn't even had it's flanges thickened but it's still more than four times stiffer !!

Engineers call the horizontal top and bottom plates "Flanges"

(the vertical plate in between is called the "Web")

A factor of x^4 gives a really astounding rise in stiffness (or decrease in sag) as you make the beam bigger. It means that if you double the size you get **sixteen** times the stiffness (which means only $\frac{1}{16}$th as much deflection. If we make the beam cross section three times bigger then we get a staggering **81**–fold boost in stiffness (3 X 3 X 3 X 3 = 81). Even if we only blow the scale up by 10% then the beam gets nearly half as stiff again (because 1.10 X 1.10 X 1.10 X 1.10 = 1.4641).

Of course, the bigger beam is a bit heavier and more expensive because it's got more metal in it. But the cross sectional area of the beam only goes up by x^2 (it's no different to changing the size of a paint diamond design really) so the double size beam is only four times heavier despite being sixteen times stiffer. Not bad!

If we take a certain beam and scale its height up but leave the width the same then you don't get quite such a spectacular increase in stiffness, in fact the stiffness goes up with the **cube** of the stretch factor. (Yep! ..another thing x^3 is useful for!)
Mind you, stretching a beam like that makes the top and bottom flanges fatter but not the "web" in the middle. They don't usually sell beams like shaped like that.

If we keep the flanges the same thickness and just make the beam deeper by stretching the web then the stiffness is a little bit less again. But it's still a bit better than $x^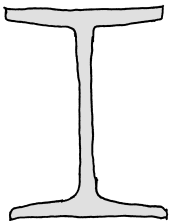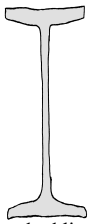2$ when stretched by a certain factor x. In fact it's slightly more complicated than that, but you don't really need to know about it —unless you fancy a quick drink in the sophisticate's lounge on the facing page.

One way of making a beam taller like this is the **castellated beam** (see top of facing page). This is where they take a standard beam and cut it in half lengthways in a zig–zag pattern. Then they weld it back together to form a deeper beam with holes in it. You might have seen these in supermarket roofs or holding up the power cables along the railway. In its most common form this increases the depth by 50%, but the stiffness more than doubles (because $1.5^2 = 2.25$) and the weight doesn't increase at all! Well worth a bit of cutting and welding!
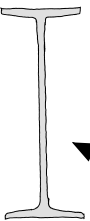
Terminology

When you multiply a number by itself and then by itself again and then by itself yet again, it's called raising the number to the power four, and it's written x^4.

You could call x^2 "x to the power 2" and x^3 "x to the power 3" if you wanted to, but nobody does (they just call them x squared and x cubed). But just like with x^2 and x^3 the little number 4 in x^4 means that x appears four times when it's written out in full:–
x X x X x X x
It's just that x^3 is the highest power to have a special name of it's own.

But using the word "power" is a bit weird in the first place. Only two pages ago we were merrily using the word to describe how much ooomph a motorcycle engine has got. And that's the usual technical meaning of the word; it's the amount of energy you're putting in per second (as in the bike engine) or getting out (as in the wind turbine). But we're using it to mean something totally different:- the number of times you multiply a number by itself. They could have invented a perfectly good new word for it, but no, they just had to make the word "power" that little bit more confusing. And turns of phrase like "girl power" and "power dressing" just add to the problem!

Did you say yes, Derek's beams will do?

It's the right answer. Derek's beams are bigger by a factor of 2 so they'll be $2^4 = 16$ times as stiff. That means that we'll only get one sixteenth the sag for the same load of dancers. Now one sixteenth of 200mm is 12.5mm. That's about half an inch, and comfortably less than the 2cm limit. Well done!

Did you say "No!, Derek's beams are dodgy"

Well that's the **wrong** answer (judging purely by their size at any rate). Maybe you thought that beams of twice the cross-sectional size would be twice as stiff. In that case the sag would be halved to 100mm which is still too much. But in fact the stiffness goes up by 2^4 (see right answer opposite). Mind you, I still wouldn't pay him until I'd seen them.

Castellated beam

The original beam is cut in a zig–zag like this...   ...and then welded back together again like this

A quick drink in the sophisticate's lounge

I said that this kind of beam does a "**bit better**" than x^2 in terms of getting stiffer as you make it deeper. In fact the proper answer is a bit of x^2 plus a little bit of x^3. When you have to work out a solution as a mixture of different powers like this it's called a "**polynomial**". And if you're astounded by how many uses we've already found for simple powers then you'd be staggered by how often polynomials crop up! However, this is really beyond the scope of this book.

I was still struggling to get my head around x^4, actually!
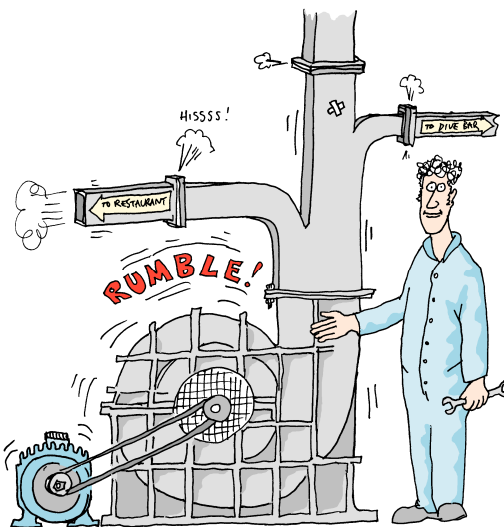
But I know what you're thinking! …you're saying to yourself "blimey!, if x^4 exists then where does it all end? …is there such a thing as x^5 or even x^6 and if so, are they useful for anything?" Well I'm glad you asked[see note i] because here I am down in the basement doing a spot of routine maintenance on the big fan that pumps air through the ventilation system to all parts of the club. Now if you enlarge one of these fans by a certain amount x, guess how much your electricity bill increases? Well knock me down with a feather and call me Nancy if it isn't a factor of x^5. That's $x \times x \times x \times x \times x$ (or " x to the power 5" or "the fifth power of x" –if you want to say it out loud). So if you triple the size of the fan its power consumption will jump by a factor of $3 \times 3 \times 3 \times 3 \times 3 = 243$!!. This is all assuming that you turn the bigger fan at the same speed[see note ii] and that you scale it up by x in all directions simultaneously; width, height **and** depth. It also assumes that both fans are the same shape and design of course.

Your biggest fan (no, really)
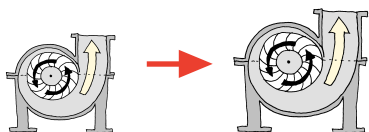
All right then, here's a problem. If we make the trusty old ventilation fan 20% bigger (all round) then how much more will it cost to run? (hint: 20% bigger means a factor of 1.2)

Did you get 6 times the original cost?

No, it's not **that** bad. …sounds like you took the increase in size (a factor of 1.2) and multiplied it by 5 instead of raising it to the power 5. So you got $1.2 \times 5 = 6.0$ instead of the correct answer, $1.2 \times 1.2 \times 1.2 \times 1.2 \times 1.2 = 2.488$

Did you get 2.488 times the original cost?

That's right. Bumping the fan size up by 20% is the same thing as multiplying it by a factor 1.2 So the electricity consumption of the motor will go up by a factor of $1.2^5 = 2.48832$…nearly two and a half times as much! –Nasty!

Note i: Ok then, I admit, you didn't

Note ii: In real life big fans nearly always turn slower, for this and other good reasons

Passwords and interest.

Well it's early in the morning and all the clubbers have gone home. But here I am still sat at the computer having a quick look at the sophisticates' lounge members accounts. You see unlike most things in the club (where you have to pay up front) the sophisticates get an account which they can charge their bills to and then pay it off later. In fact there are three members logged in right now over the internet, checking their accounts or paying them off or maybe booking some tickets for one of our special shows. Of course each member needs a secret password to log in like this. Otherwise everyone would be able to see everyone else's account. That wouldn't be very good for privacy. It could even lead to mischievous booking of phantom lobster dinners in the restaurant (which would make Chef even more cross than he normally is). There would also be nothing to stop an unscrupulous person booking a great evening out at the club at someone else's expense and then disappearing at the end of the night leaving some poor sophisticate with a large and unexplained bill. Well there are about 100 members of the sophisticates' lounge. So you might say "why not allocate each one a number 0–100 that they have to type in so that they arrive at the right account and no-one else's. But that would be very easy to abuse for a criminally minded clubber. It wouldn't take long to try out a few numbers before he got one that would let him start spending somebody else's money. Besides, people like to pick their own passwords so that they're easy to remember. So what we really need is a **huge** number of possible passwords most of which aren't actually used. This makes it hard for villains to use other peoples accounts. It also makes it unlikely that two different legitimate members will pick the same one[see note i]. This is why we use **six** character passwords for the sophisticates' lounge members accounts.
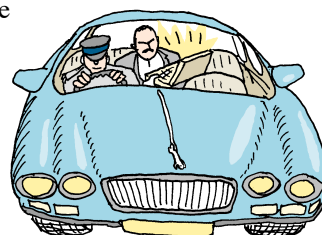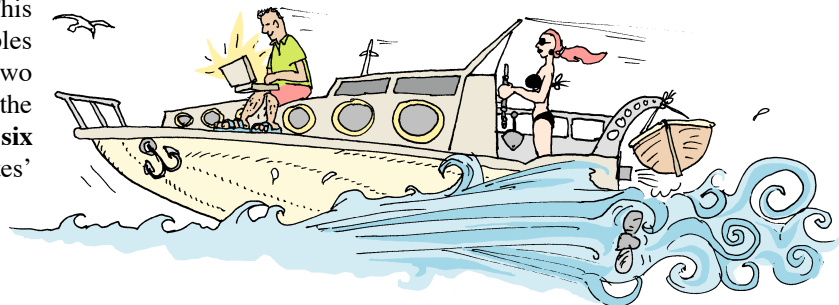
Note i: …unless they choose something too obvious.

The wonderful y^x button

On some calculators it's called the x^y button (but it does the same thing)

Here are a couple of alternative forms of the y^x button you might see

you might even see ones like this. They don't look anything like y^x, but rest assured; they do exactly the same thing.

On your calculator

Well although most calculators have an x^2 key (and a few posh ones have an x^3 key) you certainly aren't going to find an x^6 or x^7 button on there. And whereas doing $x \times x \times x \times x$ by hand wasn't too painful (remember the dancefloor beam example?) doing x^{23} most certainly would be. And working out x^{23} is exactly what you've got to do if you want to work out the number of different 23 character passwords there are, as we will see.

But luckily you don't have to! Any calculator worthy of the name will have the utterly brilliant y^x button somewhere on it. All you do is type in the number you want to raise to a power (like 26 in this case –this number is sometimes called the "base") Then you press the y^x button followed by the power you want to raise the base to. For example type 3 if you want x cubed, 6 if you want x^6 …and so on (this number is sometimes called the "exponent"). Then just press = and there's your answer! Actually this is just part of what the y^x button can do for you[see note i] as we will shortly see. Every calculator should have one!

So the question is: "how many possible passwords are there to choose from?". This should tell us if we've got enough passwords to make the system reasonably secure. Well let's start off by thinking about a **one** character password. This would be daft, but bear with me. Now assuming that we only allow normal letters of the alphabet (and take no notice whether or not they're capitals) then this will give us 26 possible passwords. That's obviously pretty useless– it's not even enough to go round the 100 sophisticates! So let's add another character and make it a two digit password. Well there are 26 different two letter passwords beginning with a; "aa", "ab", "ac" and so on. There are also 26 two letter passwords beginning with "b": "ba", "bb", "bc" etc. In fact for each of the 26 possible choices of first letter there are 26 different choices of last letter. So you can see that the total number of possibilities is 26 × 26 (in other words 26^2) which equals 676. That's a bit more like it (although it still wouldn't take very many tries for some scheming scoundrel to guess someone else's password).

All 2 letter passwords beginning with "a" → aa ab ac ad . . . az

All 2 letter passwords beginning with "b" → ba bb bc .

All 2 letter passwords beginning with "c" → ca cb cc

(etc)

. . . (etc)

. . . (etc)

All 2 letter passwords beginning with "z" → za zb zc . . zy zz
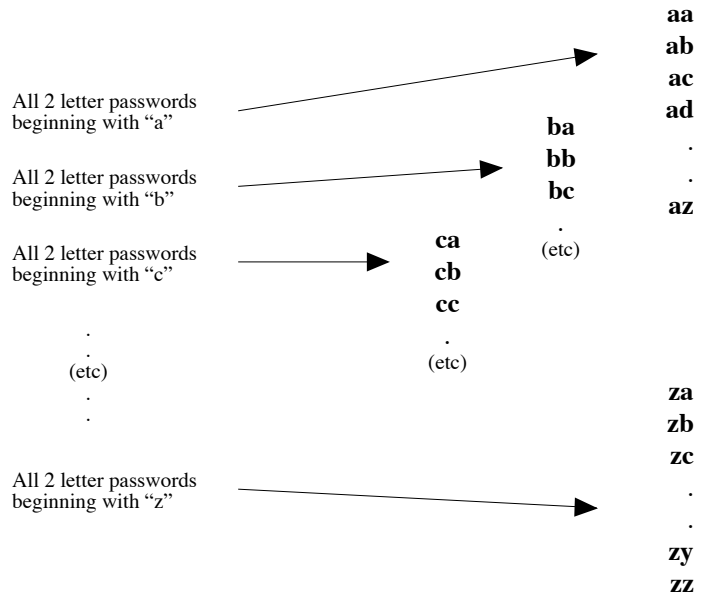
So if there's 676 combinations of two letters, then that must mean that there are 676 **three** letter passwords beginning with "a", and another 676 beginning with "b" etc. So there are $17576 = 676 \times 26 = 26 \times 26 \times 26 = 26^3$ three letter passwords in all. In fact every time we add an extra letter we just multiply the previous number of passwords by 26. So the number of different passwords of a certain length x is just 26 raised to the power x …Astounding!!

Note i: Did I mention that I quite like the y^x button? —*Captain*

Question: **How many different sophisticate's passwords are there?**

The passwords are made up of six letters. You can carry on assuming that each of the six letters is simply one of the 26 characters A–Z (because we treat "a" the same as "A" and disallow the numerical digits 0–9). An approximate answer within 1% or so will do.

In effect we're using a formula; $x = 26^n$ to work out the number of passwords of a given length n. Of course the "26" in this formula comes from fact that there are 26 characters to choose from. So there's a more general formula $x = m^n$ for working out the number passwords of length n when you've got m different characters to choose from. For example, if we change our minds and allow numbers in the passwords and make the letters case–sensitive then in addition to the letters a–z we will also have the numerals 0–9 and the capital letters A–Z as possibilities for each character. That's a choice of 62 different symbols available. So the formula becomes $x = 62^n$. So the number of six–digit passwords would leap up even further to 62^6 which is 56800235584 (and it would have been even more if we'd allowed punctuation characters like ;?()! in there too).

A healthy interest?

But look! I'm not the only one who's still here working in the early hours! Here's Ceris also tapping away at the computer. Now Ceris is just doing the accounts and stuff which basically consists of kind of… well it's sort of like errrrm… well actually I don't really understand what it consists of, so I'd better let Ceris tell you herself.

       "Thank you Captain. Well I've just been updating our records with tonight's takings and now I'm transferring some money between accounts. You see sometimes[see note i] we end up with a surplus of money in our ordinary bank account that we won't be needing to spend again straight away. When that happens I move it into a different account that pays a decent rate of interest. That way we get a bit of extra money at the end of the year, depending on what the interest rate is. Now this account I'm looking at here has a 10% interest rate, so if by some miracle we managed to keep £1000 pounds in there all year then they'd give us an extra £100 at the end. Hooray. So it's exactly the same thing as multiplying our original amount by a factor of 1.1 and we'd end up with £1100. And if by some even bigger miracle I kept it all in there the following year they'd give me 10% again only this time it'd be 10% of £1100 (not 10% of £1000) so I'd get £110 interest. So at the end of two years I'd have £1210 which is 1.1 times greater than the £1100 I had after the first year. Now if there was a totally enormous miracle and we just kept the money in there[see note ii] then you can probably see what would happen.

Did you get about 309 million ?

That's correct. The number of possibilities is 26 raised to the power 6. That's 308915776 to be exact (so there are about 3 million "spare" passwords for every member). It would take a very lucky villain to guess one first time!

Did you get 3.0891e+08?

(or something like it?)

…Actually that's also correct. In fact it's really the same as the 309 million one opposite. The strange notation is just your calculator's way of telling you it hasn't got enough digits to display such a huge number.

Did you get 156?

That's what you'd get by **multiplying** 26 by 6 instead of raising 26 to the power 6.

Nooooooooooooooooooooooooooooooo!!!!!

Note i: very very very rarely!

Note ii: This would basically involve not telling the Captain that the account existed!

Unlimited power

All the examples we've seen up to this point have involved numbers raised to one specific power. For example there was the wind pressure on the roof sign which rose with the square of wind speed. And there was the ventilation fan in the basement whose running cost grew with the fifth power of it's size. In both examples we multiply a number by itself a particular number of times because that's what the physics of the situation dictates.

But in the examples in this chapter we raise numbers to arbitrarily high powers. There's essentially no limit to the exponents we might use. If we want to know how secure a 23 word password is then we just go right ahead and raise some number to the power 23. Or if we want to know how large some fund will have grown after 47 years then some number is going to get raised to the 47th power.
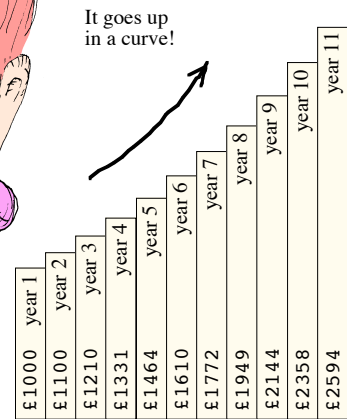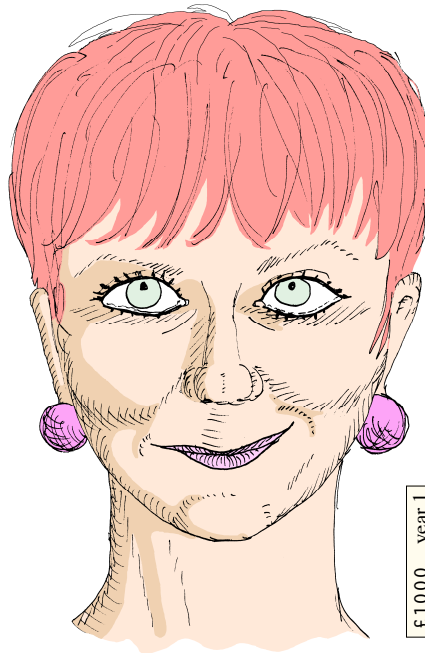
An interest problem

Give this one a try: If you have £500 and you leave it in an account paying 17% interest for 9 years, how much money do you end up with?

Every year I would get some interest but every year it would be a bit more than the last. So the bank balance would keep increasing but it wouldn't be at a constant rate; it would actually go up in a curve. Some people call this phenomenon the **miracle of compound interest**.

I often need to work out how much the money will grow after a certain period of time at a certain interest rate. For one year it's quite easy; a 10% interest rate means it grows by a factor of 1.1 and so on for other interest rates. Over two years it grows by 1.1 and then by a factor of 1.1 again so we end up with $1.1 \times 1.1 = 1.21$ times as much money (which is 1.1 squared). In fact whenever we keep the money in for an extra year it just multiplies the final balance by 1.1 again (or by a different number for different interest rates). In fact all I have to do is multiply the original amount of money by 1.1 raised to the power **n** (where **n** is the number of years). So the growth after 10 years would be 1.1^{10} which would be a factor of about 2.59. I suppose £2593 isn't bad but it's a bit of a long wait. If on the other hand the interest rate was 20% then after 10 years the growth would be a factor of 1.2^{10} which is about 6.19 —that's more like it!"

It goes up in a curve!

£1000	year 1
£1100	year 2
£1210	year 3
£1331	year 4
£1464	year 5
£1610	year 6
£1772	year 7
£1949	year 8
£2144	year 9
£2358	year 10
£2594	year 11

a) Did you get £2054 and 20p?

Absolutely correct. The money grows by a factor of 1.17 to the power 9 which equals 4.1084. All we have to do is multiply our original £500 by this factor.

b) Did you get £1265?

Incorrect. This is the amount you'd get by adding 17% of your £500 (= £85) for each of the nine years. But we are adding 17% of the **total amount so far** each year. It makes a big difference!

c) Did you get £592929382485007

Sadly not! I think you multiplied your money by a factor of 17^9 when you really meant $(1.17)^9$. Your answer would be correct for an interest rate of 1600% (very nice if you can find it!)

A graph of
Captain's
personal
account

Thanks for that Ceris! I think I understand it a bit better now. In fact I've just been looking at my personal account and it's going in a curve just like you said! Do you think I'll soon have enough money to trade in my 500cc Nightclub Ninja for a 1000cc Super Bass Burner Special like Doug's??

"No you idiot! Your account is going in a curve all right but it's curving **downwards**. Since you had an overdraft to begin with (which is like a negative sum of money) you'll get a negative amount of interest each year. In other words you'll have to **pay** interest. And if you don't pay any of the overdraft off then each year the amount you'll pay will be bigger. So the debt will get bigger at an accelerating rate. Actually, it's even worse than that, because banks invariably **take** higher rates for overdraft interest than they **give** for savings account interest. So the curve going downwards is steeper than the one going up. This is the reason[see note i] why banks get rich; they use the mon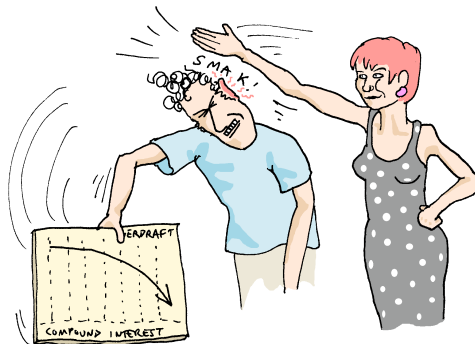ey in savings accounts like mine and lend it out as overdrafts to gullible people like you at a higher rate. Then they just sit back and let the profits accumulate without having to manufacture any goods or run any nightclubs or anything! And I haven't even mentioned the dreaded 'charges' yet! Don't get me started on that topic!"

The powers of 10

Actually, there's an application of raising a number to an arbitrarily high power that we use every day without even thinking about it. I'm talking about our ordinary base 10 system of numbers. You're so familiar with it that you forget that there were ever any other ways of writing numbers down (like Roman numerals or cutting notches in a bone). But it's a brilliant system that can handle both tiny and huge numbers and lets you do sums and calculations with relative ease. The key idea is that each consecutive digit in a number represents the next power of ten. For example, the number 1200 means one lot of 10 cubed (=1000) plus two lots of 10 squared (=100). If we need bigger numbers we just add a column on the left for 10^4 (the ten thousands) and maybe another one for 10^5 (the hundred-thousands) and so on. we never run out of numbers, –unlike the Romans! A complete explanation of Roman numerals is beyond the scope of this book[see note ii] but suffice it to say that they used different symbols to represent increasingly large quantities. There was V for the fives, X for the tens, L for the fifties, C for the hundreds and so on. They just wrote these symbols down enough times to get the right number. For example, 235 in Roman would have been CCXXXV (two lots of C, three Xs and a V). Now these "numbers" looked very imposing when carved into stone tablets, but just imagine the aggravation when they ran out of numbers! Instead of just adding another column on the left like we do they had to phone everybody up in the entire Roman empire and agree on a new symbol! This was extremely inconvenient because they hadn't invented the telephone. Actually, that was just one of many problems with Roman numerals (imagine trying to multiply or divide them!) For more information see the (excellent) book in note ii.

Note i: Well, **one** of them!

Note ii: For the full story about Roman numerals (and hundreds of other kinds of number) read Georges Ifrah's excellent book "The universal history of numbers" —reference 1 in the bibliography.

The great kitchen telegraph

On the binary system, and sums of powers

This chaotic scene is absolutely typical of a busy night in the kitchen. At least, that's how it **used** to be. You see the staff up in the restaurant used to telephone the orders down to Chef, but the kitchen is such a noisy place that Chef often couldn't hear them properly. This not only made Chef **even angrier** but it also led to confused orders and disgruntled customers. They tried sending down little paper notes in the dumb waiter but that was even worse (they got blown down the cracks and ended up in the basement). That was when chef decided to build his **Great Kitchen Telegraph**. We had a few teething troubles with the early models but in the end it worked astoundingly well and we've been using it ever since.
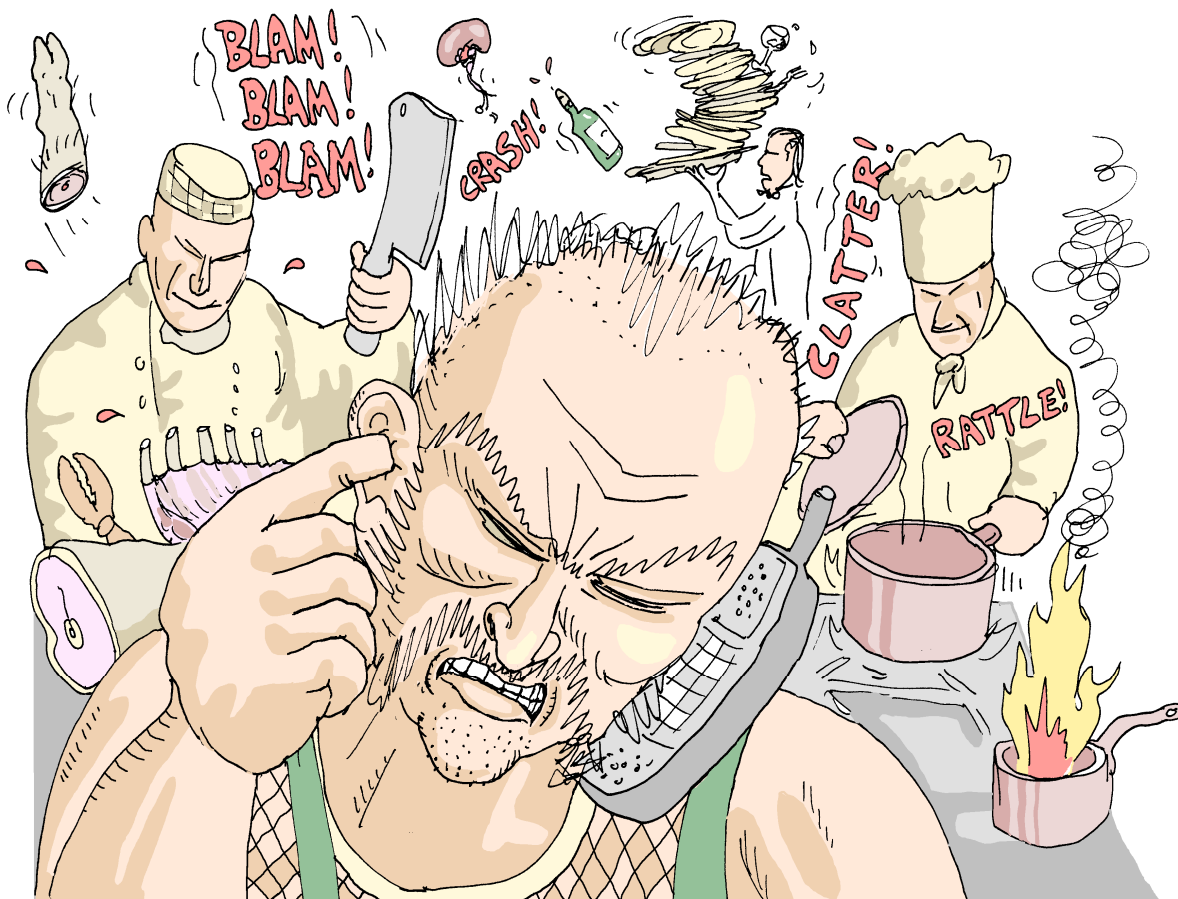
The Mk I (how it works)

Well it's not rocket science. The Restaurant staff upstairs pull a lever which pulls a string which flips up a little flag in the kitchen. Each flag has the name of a particular dish written on it which Chef can see at a glance –no matter how noisy the kitchen gets!

Waiter in restaurant pulls lever

Overworked waiter

Strings run all the way down to the kitchen

Chef's cat

There's a separate string for every dish

Chef eagerly awaits incoming orders
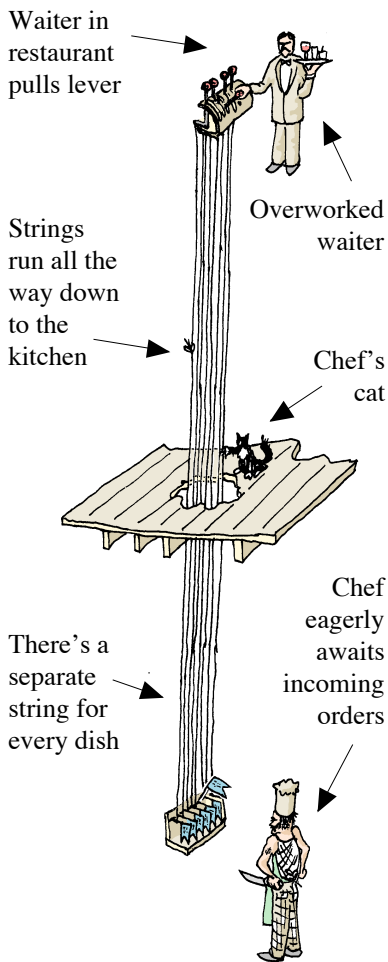
This worked very well for the dive bar where we only serve about half a dozen simple and inexpensive (but tasty) dishes.
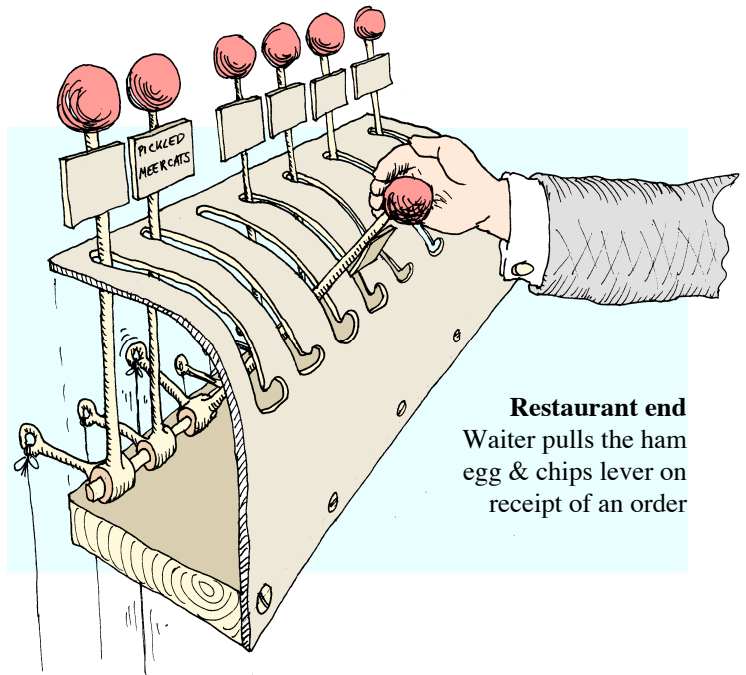
Restaurant end
Waiter pulls the ham egg & chips lever on receipt of an order

The weakness in this design is the fact that you need a new string every time you add a new dish to the menu. It was all right for the dive bar where we only serve a few different snacks but for the restaurant (and especially for the sophisticate's lounge with it's seven–course dinners) we began to run into problems.

HAM, EGG & CHIPS

PICKLED MEERCATS
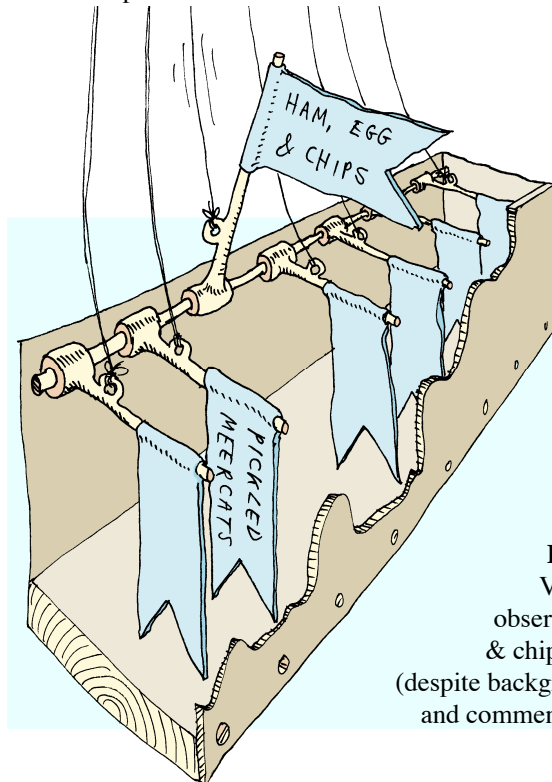
Kitchen end
Vigilant Chef observes ham egg & chips flag rising (despite background noise) and commences cooking
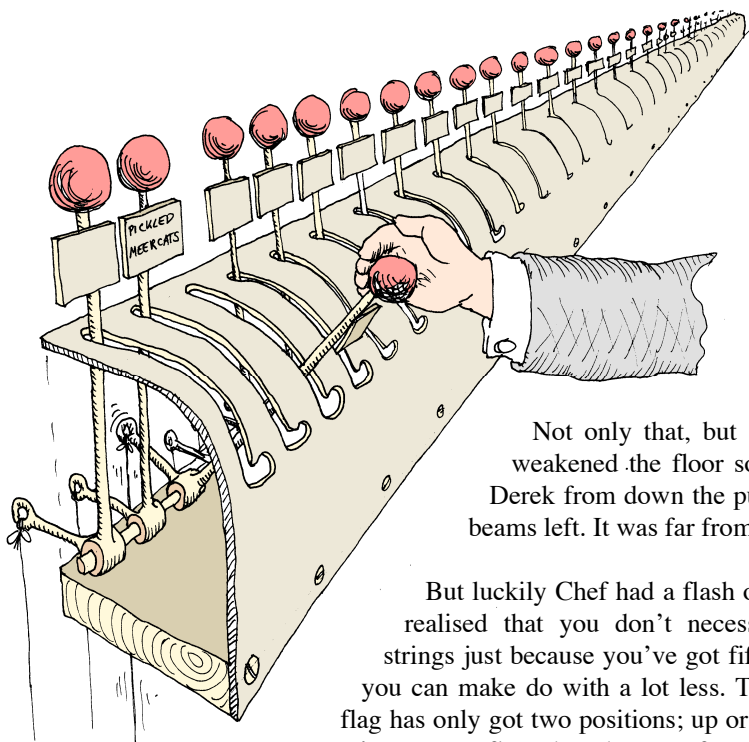
The Mk II   (...too many strings!)

This is the unfortunate result of extending the system to cope with with the extensive menus of the main restaurant and sophisticates' lounge. What was convenient for half a dozen items has become a bit of a nightmare! Chef went cross–eyed trying to watch fifty flags at once and the waiters exhausted themselves frantically running up and down the row of levers.

Not only that, but all the holes we drilled for the strings have weakened the floor so seriously that we had to phone Phil's mate Derek from down the pub to see if he'd got any more of those cheap beams left. It was far from ideal.

But luckily Chef had a flash of inspiration. He realised that you don't necessarily need fifty strings just because you've got fifty dishes. In fact you can make do with a lot less. The thing is, each flag has only got two positions; up or down. But when you've got two flags then there are four possible patterns; the first flag can be either up or down while the seconds flag is **up**, and it can also be either up or down when the second flag is **down**.

1)   2)   3)   4)

Now if we add a third string then the first two flags can be in any of these four patterns while the third flag is **up**, and they can also be in any of those four patterns when the third flag is **down**. So adding the third string simply doubled the number of patterns to eight. In fact, whenever we add an extra string we double the number of flag patterns! To make use of this phenomenon the restaurant and kitchen have to agree on a code[see note i] whereby each dish is assigned to it's own unique flag pattern. Then we can handle loads of different dishes with a considerable saving in string!
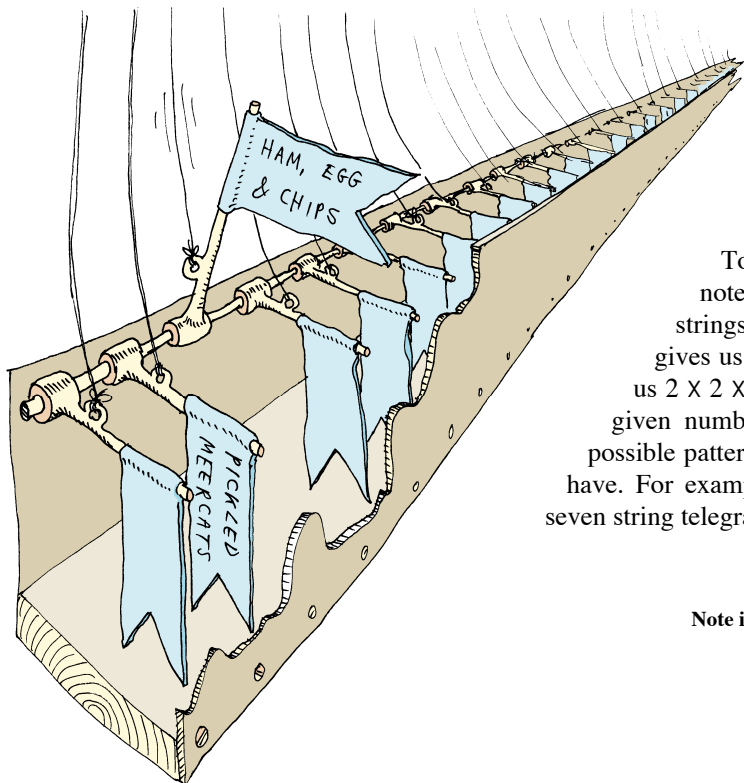
To work out how many strings are needed we note that one string gives us 2 choices, 2 strings gives us $2 \times 2 = 2^2 = 4$ choices, 3 strings gives us $2 \times 2 \times 2 = 2^3 = 8$ choices, 4 strings gives us $2 \times 2 \times 2 \times 2 = 2^4 = 16$ and so on. In fact for any given number of strings x there are going to be 2^x possible patterns and therefore 2^x different dishes we can have. For example that's 128 choices[see note ii] from a mere seven string telegraph. Quite astounding really!

Note i: ...dictated by Chef, actually.

Note ii: In practice we reserve one of the patterns to mean "nothing" or "off". So if you're being picky you might say that "nothing" doesn't really count as a choice and there are only 127 real options.

The Mk III (binary model)

This is the highly successful MkIII telegraph system. The main difference is that we now read the dishes from the chart, not directly from the little flags.

Restaurant copy of the flag chart

Just a few strings!

Happier waiter

Chef's cat

Chef's copy of the flag chart

Dish names no longer written on flags
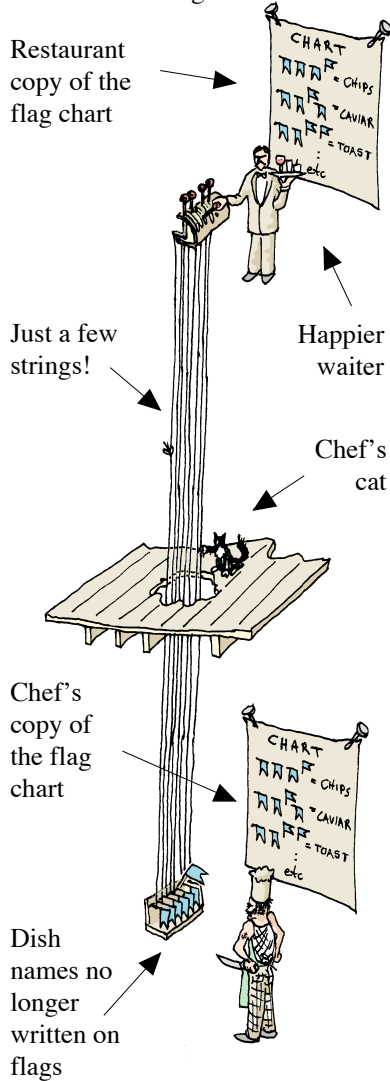
A question

Have a go at this one: How many different dishes could you handle on an eleven string version of this telegraph?

The key to this system is agreeing a common code (see chart opposite) which tells you which dish each pattern of flags represents. If the restaurant chart isn't exactly the same as the kitchen chart then the system goes horribly wrong! But once you've learned the right patterns it's a convenient system which tells you the dish wanted out of hundreds by a quick glance at a few flags. It's also admirably economical on string and doesn't weaken the floor.

The BINARY system.

The improved telegraph is an example of the binary system. Each signal is made from a pattern of flags each of which can be in only one of two positions, up or down. Well the memory in your mobile phone that stores all your favourite numbers works in exactly the same way. The memory itself works just like a row of switches each of which has only two states, on or off. These two states sometimes get called true and false or 0 and 1 (even though they're not really numbers). Just like Chef's telegraph you rely on an agreed standard system for deciphering a particular pattern of 0s and 1s into the number that it really means. Also like the telegraph we get 2^x different possibilities for x strings. The only difference here is that we're not talking about strings but "bits" (as

```
000000 = 0
000001 = 1
000010 = 2
000011 = 3
000100 = 4
000101 = 5
000110 = 6
000111 = 7
001000 = 8
     :
   etc
```

these electronic switches are known). Well computer memory is basically the same stuff as the memory in mobile phones[see note i] although there's usually more of it. So it's no coincidence that you see a lot of numbers like 256 and 512 in computer advertisements because they're powers of two ($256 = 2^8$ and $512 = 2^9$).

Did you get 2048?

That's correct. It's 2 raised to the power 11. I hope you've got a y^x button on your calculator because otherwise that must have been very tedious to compute.

Did you get 2047?

Actually that's even more correct if anything. It's equal to $2^{11} - 1$ which is all the combinations minus one. This is good because it leaves one pattern (ie all flags down) to mean "no orders".

Note i: There are different types of memory, but they nearly all work like this

CHART

⚑⚑⚑⚑⚑ = CHIPS

⚑⚑⚑⚑⚑ = CAVIAR

⚑⚑⚑⚑⚑ = TOAST

⚑⚑⚑⚑⚑ = ALL DAY BREAKFAST

⚑⚑⚑⚑⚑ = PICKLED MEERCATS

⚑⚑⚑⚑⚑ = HAM, EGG & CHIPS

⚑⚑⚑⚑⚑ =

⚑⚑⚑⚑⚑ =
⋮

etc
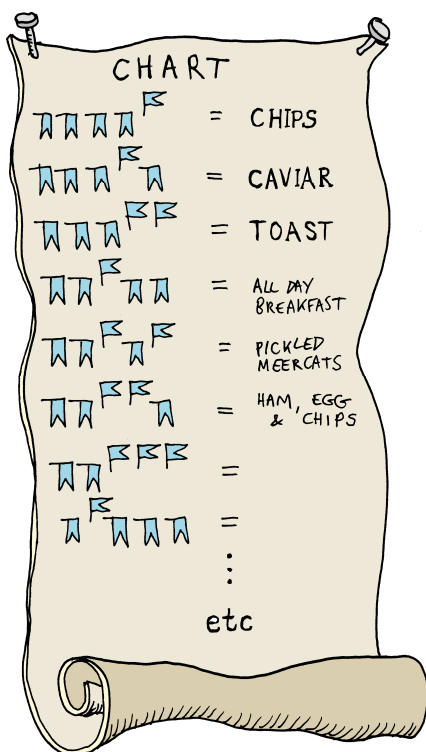
Chef's Law $x^a \times x^b = x^{a+b}$

In the course of his experiments Chef has noticed a mildly astounding thing. He's been cannibalising two of his earlier smaller models to put together and make one big telegraph. Now let's say you take your existing telegraph and extend it by glueing on an old five–string telegraph on the side (like Chef's doing below). Well the five string telegraph has 32 different combinations on it. So if you pick any single combination of flags on the original telegraph and hold that still you'll get 32 new combinations just by waggling the new flags through all their possible patterns. In other words, however many dishes you could signal on your original telegraph, you're going to be able to handle 32 times as many on the new combined telegraph. In fact, to be a bit more general about it, whenever you combine two telegraphs together like this then the number of dif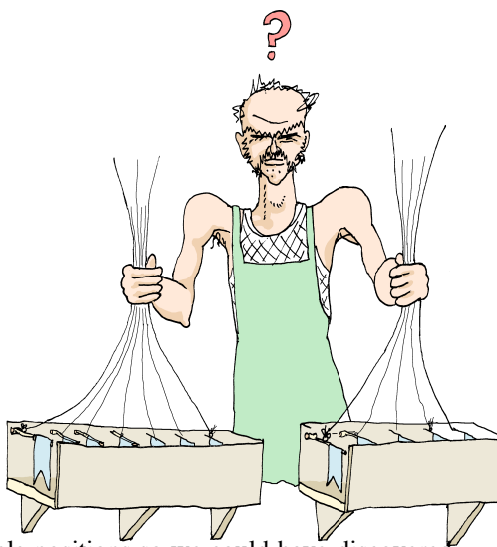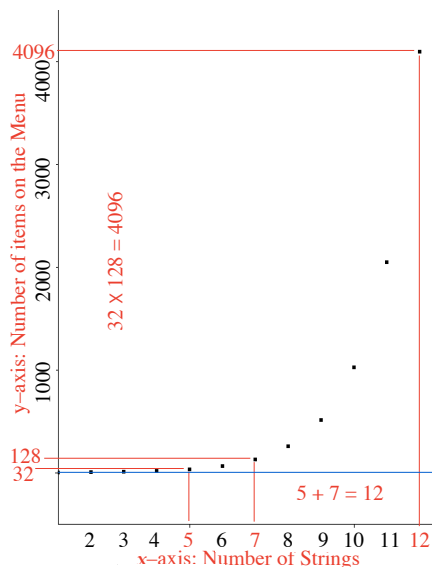ferent signals that the resulting telegraph can do is equal to the numbers that the old ones could manage multiplied together. For example, here is Chef adding a five string (32 dish) telegraph on to the end of his old seven string (128 dish) telegraph. So he's going to end up with a humungous telegraph capable of handling 32 × 128 = 4096 different menu items! But there's a totally different way he could have worked it out (and this is the astounding bit). The thing is, when Chef combined those two old telegraphs he was really just scavenging as many strings as possible for the new one, and he ended up with twelve. Now we already know that any n string telegraph is going to have 2^n possible positions so we could have discovered the new model's 4096 dish capacity by simply adding up the strings. We didn't actually need to know the capacities of the old machines to multiply together. Well as you can see, I've plotted a graph of number of strings (on the x axis) against number of menus items (on the y axis). There's a nice curve with y starting to whiz upwards really quickly as you add more strings. And you can see the interesting property that adding any two numbers on the x axis has the same result as multiplying the corresponding numbers on the y axis. It works for any two numbers[see note i], not just this particular example of 5 + 7 = 12 (therefore 32 × 128 = 4096). But Chef wonders whether this is just a fluke that works with the powers of two, or whether it might still work with 3^x or 5^x and so on[see note ii].

Graph:
y–axis: Number of items on the Menu

4096 ···
32 × 128 = 4096
4000
3000
2000
1000
128
32

5 + 7 = 12

2 3 4 5 6 7 8 9 10 11 12
x–axis: Number of Strings

Reader's voice

Oi! Captain! …why haven't you drawn a nice curve running through the points on that graph?

Well, …I wasn't really sure what the parts of the curve in between the points would **mean**. I can work out what 2^3 or 2^4 are but is there such a thing as $2^{3.5}$? …after all, you can't really have a three and a half string telegraph can you?
I've got a feeling we'll be hearing more about this later on.

Note i: In other words $2^a \times 2^b = 2^{(a+b)}$ no matter what values you choose for a and b. Our example of $2^5 \times 2^7 = 2^{12} = 4096$ is just one special case of this general rule.

Note ii: ie: Chef is wondering if the rule might be even more general than $2^a \times 2^b = 2^{(a+b)}$. Maybe it would work with powers of **any** number $c^a \times c^b = c^{(a+b)}$

MkIV   the ultimate model

Due to sporadic string breakage, cat damage, and other problems, Chef finally developed the MkIV which employs sticks instead of strings.

3"

3"

Ordinary 3" square timber is used for the sticks

Waiter rotates sticks

Cat–proof design

Each of the 4 sides of each stick carries a different symbol

Chef observes the pattern of symbols.

Both restaurant and kitchen are issued with a different kind of chart for the new telegraph.

Top bearing

Support bracket

Restaurant end

Comfortable hand–wheels (for ease of stick–turning)

The four card suits; hearts, clubs, diamonds and spades are used as the symbols for the four positions of each stick. The waiter rotates the sticks until the right symbol is facing the front. Each dish has it's own pattern of suits (instead of the pattern of flags we used before).

Kitchen end

Bottom bearing

Each stick is numbered and has a pointer for easy symbol reading

CHART

♣ ♣ ♣   = CHIPS
♣ ♣ ♥   = CAVIAR
♣ ♣ ♠   = TOAST
♣ ♣ ♦   = ALL DAY BREAKFAST
♣ ♥ ♣   = PICKLED MEERCATS
♣ ♥ ♥   = HAM, EGG & CHIPS
♣ ♥ ♠   =
♣ ♥ ♦   =
♣ ♠ ♣   ⋮
        etc
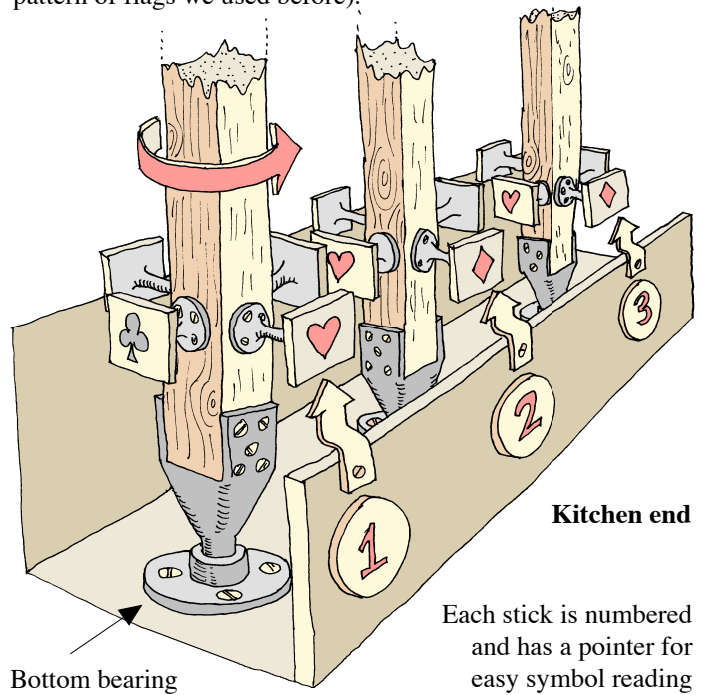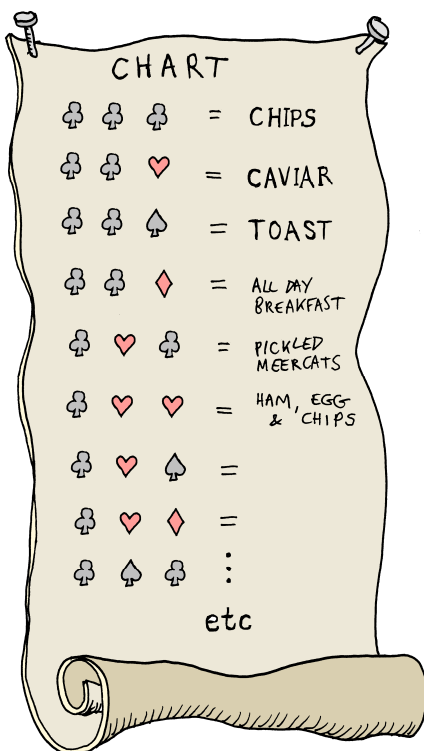
…and here's the new chart for the MkIV. Note that it starts off by stepping through all four symbols on the first stick with the others all held on clubs. Then we increment the second stick to it's next position (hearts) and step through all the first stick symbols again etc etc. So it's just like the MkII flag telegraph really except that there are **four** positions per stick instead of **two** positions per string. This means that a single stick telegraph would be capable of handling four different dishes, and every time we add a new stick to the telegraph we **quadruple** number of menu items.

Question   What's the capacity of the MkIV?

Here's a tricky one for you. How many menu items can you handle on a MkIV stick based telegraph? I'm not talking about a **specific** telegraph with a certain known number of sticks. I want the **general** answer for any MkIV telegraph with any number of sticks (let's call that number x[see note i])

Well I've plotted out a graph showing how your menu capacity improves as you add sticks to a MkIV telegraph and you can see that it's similar to the curve you get with the string telegraph, only it goes up even more sharply as sticks are added[see note ii]

And just like the MkIII string telegraph curve ($y = 2^x$) there's this strange property that **adding** two numbers together on the x axis means **multiplying** them on the y axis
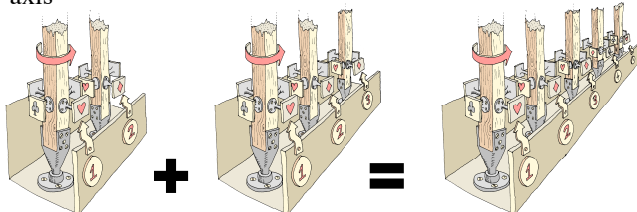
For example, if we add a 2 stick to a 3 stick telegraph then the resulting 5 stick device will handle $4^5 = 1024$ items. That's the same result as **multiplying** $4^2 = 16$ by $4^3 = 64$, which are the capacities of the two original separate telegraphs

Maximum number of items on Menu

$16 \times 64 = 1024$

$2 + 3 = 5$
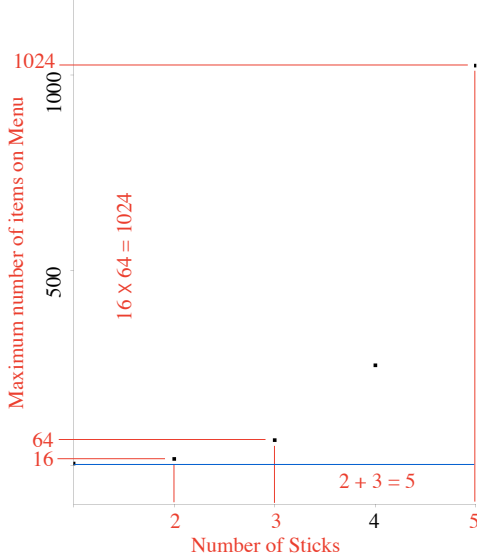
Number of Sticks

Did you get 4^x?

Well it's the right answer! Quadrupling the number of dishes with each extra stick means you get 4 choices on one stick (=4^1), 16 choices with two sticks (=$4\times4=4^2$),64 choices with three sticks (=$4\times4\times4=4^3$) and so on. So it's 4^x choices for a telegraph with x sticks!

Did you get x^4?

That's the wrong answer. x^4 just means $xxxxx$ but we need to do an additional multiplication (by 4) for each additional stick. For example, a 5 stick MkIV will handle $4\times4\times4\times4\times4 = 5\times5\times5\times5\times5$, not 1024. $4^5 = 1024$ dishes, not $5^4 = 625$ dishes!

Did you get $4x$?

This is not correct. It would mean that we were only **adding** 4 new menu possibilities with each stick. But in real life adding a new stick gives us 4 new patterns for every single pattern that we had before. So we have to **multiply** the previous number by 4.

Note i: …how original!

Note ii: …Check out how the x and y axes are labelled. (we've had to stretch the y axis to fit it in!)

n^x …that's the number of combinations you'll get on a telegraph with x sticks when there's n positions on each stick. For example our original Mk IV with three four–sided sticks gave us $4^3 = 64$ different options. If we'd used four six–sided sticks then we'd have ended up with $6^4 = 1296$ different possibilities.

The single stick telegraph

If I asked you how many options we could handle on a telegraph with only one stick then you probably wouldn't bother with the formula n^x. It's obviously just going to be the same as the number of sides on the stick. But if you **did** use your calculator to work out 4^1 (for the 4–stick case) then you'd get exactly the same answer. That's because any number raised to the power 1 is just equal to itself. $x^1 = x$

ZERO stick telegraphs ?!

And while we're thinking about telegraphs with a very small number of sticks we might as well go the whole hog and consider a telegraph with **no sticks at all!** Now you might think that a restaurant with a zero stick telegraph wouldn't be able to have any menu items at all. But that's not true. If there was only **one** item on offer then no signalling system would be needed because we already know what everyone is going to eat. So zero sticks will do fine. And sure enough if you raise any number you like to the power 0 on your calculator, 1 is the answer you'll get! because $x^0 = 1$

Adding powers with Chef's law $x^a \times x^b = x^{a+b}$

In fact you can draw the graph of any number raised to a power x and you will always get the property that adding two numbers on the x axis means multiplying them on the y axis. With the Mk III string–based telegraph it was a graph of 2^x. With the Mk IV stick-based telegraph it was a graph of 4^x. But it would have worked just as well with a graph of any number n^x. With the stick telegraph we knew that extending it by bolting a spare telegraph on the end would multiply the number of menu choices by whatever number of choices that spare telegraph provided. That's not too astounding, because you can imagine working through each position on the spare telegraph with the old one fixed on any of it's previous combinations. But we could also find out how many combinations were available on the extended telegraph by just counting up the total number of sticks and doing 4^x (which also isn't incredibly astounding, because each new stick quadruples the number of choices). But the same argument would work even if you didn't use four–sided sticks. For example if you used five–sided sticks then the capacity of an x–stick telegraph would be 5^x. You wouldn't be able to use hearts, clubs, spades and diamonds any more of course, you'd have to use A,B,C,D & E or something. But it would still be true that combining two different telegraphs would give you a machine capable of handling whatever the two old ones could do multiplied together. And it would also still be true that you could just work out the number of combinations by counting sticks! (only with five sided sticks the answer would be 5^x rather than 4^x). So perhaps you can see that this "trick" actually works no matter how many sides your sticks have got, or as the mathematicians would write it:– $x^a \times x^b = x^{a+b}$

If you built a telegraph using ten sided sticks[see note i] then it would be rather like using our ordinary base ten numbers. Each side of the sticks could be labelled with the digits 0 to 9. A three stick telegraph would have a units stick, a tens stick, and a hundreds stick just like a three digit decimal number. So there would be 1000 different combinations from 000 to 999 (and of course $10^3 = 1000$). And because the powers of ten 10^x are just just written as a 1 followed by x zeroes it's quite easy to see that you can multiply any two of them by just adding up the zeroes. For example a hundred ($= 100 = 10^2$ ie two zeroes) multiplied by ten–thousand ($= 10000 = 10^4$ ie four zeroes) equals a million ($= 1000000 = 10^6$ ie six zeroes).

Note i: …Don't bother asking for five sided or ten sided sticks down at your local builder's merchant. –they might look at you in a funny way

…hurry up lads, it's almost time for the song

Song

The power
of one

Did you lose your power?
Did you lose your nerve?
You run in a straight line
But I run in a curve
We touched once for an instant
Then I was moving on
Leaving you behind me
raised to the power of one

You know you should forget me
You know you could do more
The root is just a fraction
of the power you had before
You think you know the product
You get when powers combine
But you can't change direction
*You're trapped on your straight line*

Eh?

This song is about the fact that any number raised to the power 1 is just itself So the "curve" of x^1 isn't a curve at all; it's a straight line. It also touches on the fact that any number raised to the power zero is equal to 1 (weird but true). And all the curves of x raised to a whole number do indeed "touch once for an instant". That's because they all pass through the point $(x = 1, y = 1)$[see note i] Also, the bit about power being an endless family of curves refers to the fact that it's quite possible and meaningful to raise a number to any power you like including ones like 3.5 and 1/3 which aren't whole numbers. As for the root being a "fraction of the power" , —well you'll have to wait for chapter 6 to find out what that's all about

They say there's power in knowledge
And total power corrupts
And power's an endless set of curves
That nothing interrupts
Without a higher index
With no–one else to blame
With one as an exponent
Your path remains the same

When all the beans are counted
And all the battles fought
And all the world is unity
(raised to the power of nought)
Far up the abscissa
When I have long since gone
You'll remain forever
Raised to the power of one

Note i: Actually they also all pass through the origin $(x = 0, y = 0)$. If you spotted that then please report to the sophisticates' lounge immediately for a free drink. If you don't believe it then have a look at the graphs in appendices I and II.
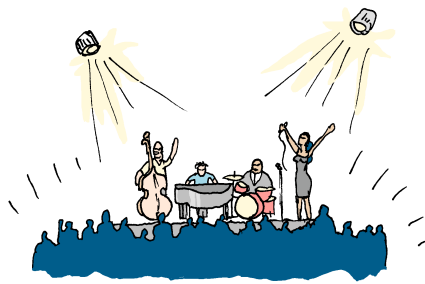
Rapturous applause

Powerful colours

**Colour vision and combinations of colours**

I love this colour digital photo of the club that I keep hanging on my office wall. It was taken by my friend Noreen during a low level flypast in her home built microlight aircraft. If you look really closely at the photo you'll see that it's made up of thousands and thousands of tiny coloured squares (or "pixels") but it just looks like a picture when you view it from a normal distance.

In fact the digital camera that Noreen uses sees colour pretty much same way that we do when we look at an object with our eyes. What I mean is that for every coloured square on the finished photo there's not one but **three** detectors: one for red, one for green and one for blue. It's pretty much the same situation in your eyeball too, (if you're prepared to overlook a few minor technical differences). What happens when you or the camera see a vivid blue object is the blue detector getting lit up 100% while the other two are hardly triggered at all. If another part of the scene you're looking at is bright red then the ray of light striking the right part of your eyeball (or the right part of the camera's CCD[see note i] sensor) will stimulate the red detector there 100% while leaving the blue and green ones quiescent. And if there's no light at all coming from a certain part of the scene then all three detectors register 0% and we perceive that bit to be black. Conversely, if all three detectors get stimulated to 100% then we see that part as white.
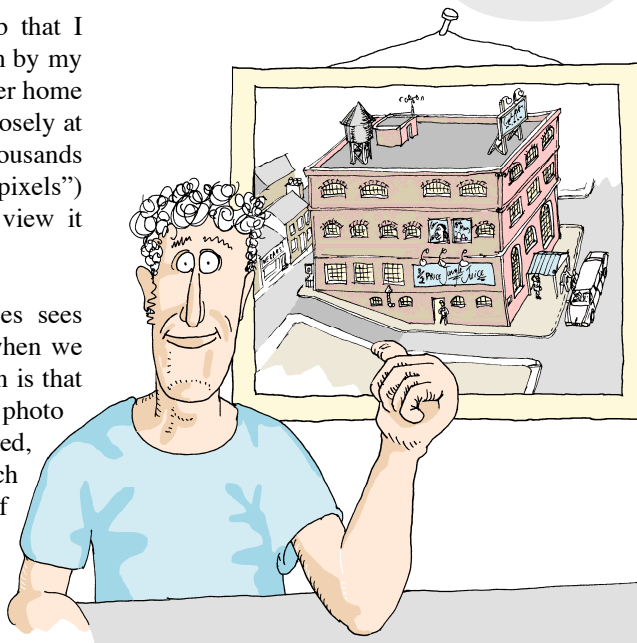
The nice thing is that it's the intermediate mixtures of different levels on the three detectors that gives us the huge range of different colours we can see. For example 50% on both green and blue with 100% on red gives you a nice salmon colour. If you have 0% red, 25% blue and 50% green on the other hand, you get a dark mossy green and so on.

Now according to Noreen, there are two key things to look out for in a digital camera. The first thing is having lots and lots of pixels, and the second thing is having lots of alternative colours available for each pixel. Having lots of pixels ensures that the picture is sharp and having lots of colours ensures that the picture colours will be true.

▭ 100%	
▭ 50%	
▭ 50%	

Salmon pixel

|	
▭ 70%	
▭ 70%	
▭ 70%	
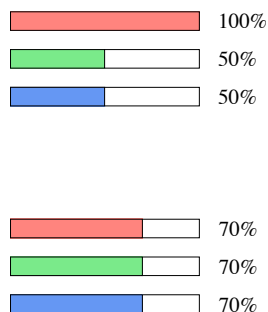
Grey pixel

Note i:CCD stands for "charge coupled device" and it's the name for a common type of electronic light detector that you get in digital cameras, scanners, and stuff like that. The detectors in your eye on the other hand are called "cones" (because of the shape of the cells) and they sit nestled together in a layer on the back of your eyeball called the "retina".

Different creatures have different levels of colour vision, some better and some poorer than ours.
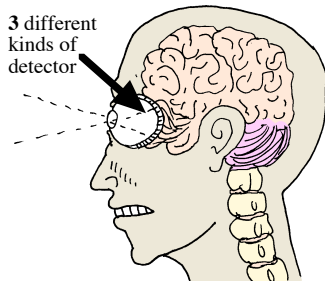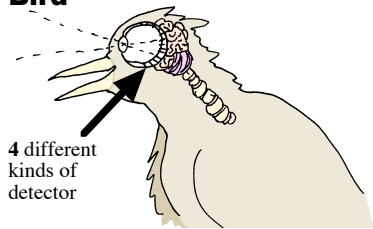
Dog

2 different kinds of detector
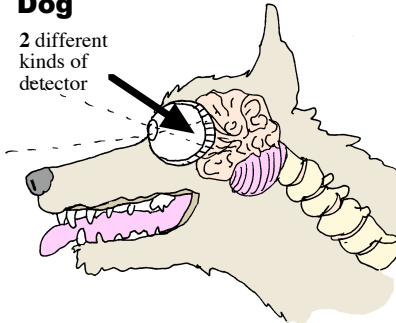
Man

3 different kinds of detector

Bird

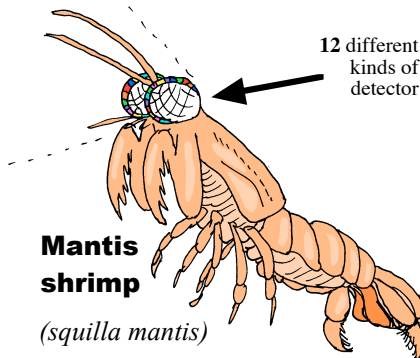**4** different kinds of detector

12 different kinds of detector

Mantis shrimp

(squilla mantis)

The incredible Mantis shrimp has no fewer than 12 different colour detector types! —Astounding!!

The digital camera Noreen uses takes a big square grid of pixels 2000 squares wide. So it's quite easy to work out how many pixels there are; it'll be 2000 X 2000 = 2000^2 = 4000000. But how many different colours can each pixel be? It turns out that Noreen's camera can distinguish between 128[see note i] different levels on each of the 3 detectors for each pixel. In other words you get a "score" of 1 to 128 from each detector depending on how much light of the right colour hits it: 1 means completely dark, 128 means maximum brightness and 64 means means about 50% etc etc. Now every single different combination of scores from red, blue and green results in a different colour in the final picture. So even if our camera was broken and the red and green detectors were stuck on 1, we'd still have 128 different colours, namely the shades of blue from pure blue through dark blue to black. If we then took the camera to a not very competent repair shop and got the green detectors un–stuck then we'd have 128 times more different colours because for every shade of blue we'd have all the possible levels of green mixed with it. So there'd be 128^2 = 16384. If we then took the camera to a really good repair shop and got the red detectors un–stuck too then there would be 127 new colours on top of each of the 16384 blue/green mixtures that we had before. A fully repaired camera would therefore have 16384 X 128 = 128^2 X 128 = 128 X 128 X 128 = 128^3 different colours!

Try this problem

Suppose you had a slightly cheaper camera than Noreen's that could only detect light levels in 10% increments from 0 to 100%. How many different colours would it be able to distinguish?

The reason why the answer was 128 to the power **three** for Noreen's camera was because of the **three** different detector types for each pixel: red green and blue. But in fact humans are a bit weird in having three different detectors; most mammals only have two. Take the humble dog for instance. He's got yellow detectors and blue detectors –but no third type. So if they made digital cameras specially for dogs[see note ii] they'd probably only put in a two channel CCD detector.

Did you get 1331?

That's right. There are 11 different levels for each detector:- 0,10,20,30,40,50,60,70,80,90,100% so the answer is $11^3 = 1331$.

Did you get 1000?

Ha! …tricked you! You probably thought that there were 10 different levels for each detector. But there are really 11 because you have to count 0% as well.

Hmmmm… This argument sounds kind of familiar!

Phil

Note i: Do you recognise that number? Well it's a power of two (128 = 2^7). And that's no accident because digital cameras store their pictures on memory (just like the computer or mobile phone memory in chapter 4)

Note ii: Don't laugh! It **could** happen!

Having only two different kinds of colour detector doesn't mean that the dog sees in black and white[see note i] (as was once thought) it just means that there are some colours that humans can tell apart that look the same to dogs. In fact red/green colour blindness in humans (which is quite common –especially in men) produces a similar effect. It means that you can see fewer colours. For example, if each detector cone cell can distinguish 100 levels of brightness (in all animals) then a human can recognise 100 X 100 X 100 = 100^3 =a million different colours. But a dog can only recognise 100 X 100 = 100^2 =ten thousand different colours.

But some animals actually have better colour vision than us. Most birds, for instance, have an extra colour detector in the ultraviolet[see note ii] range. So with four different types (still assuming 100 levels) they would be able to see $100^4 = 10^8 =$ a hundred million different colours! This means that there are many colours that look exactly the same to us but are completely different to birds. But there are some animals whose colour vision is even more astounding than birds'. Observe this plate of mantis shrimps I've just found in the kitchen. These amazing little crustaceans have not five, not six, but **twelve** different types of cone each detecting a different wavelength of light. They have eight types which respond to wavelengths in the visible range (visible to humans, that is) and another four different ones in the ultraviolet. This must give them an extraordinary level of ability to discriminate subtle colour differences. I suppose it must make them very good at looking for things that mantis shrimps like to eat. Or maybe it helps them to spot predators who want to eat them. It didn't help enough to avoid them ending up on our restaurant menu though…
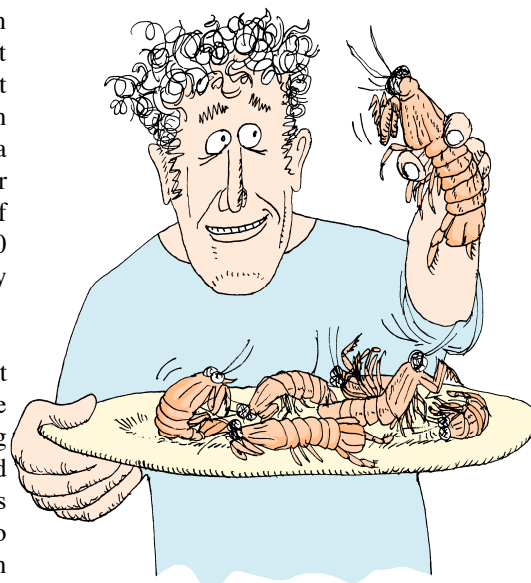
Have a go at this one

For the sake of argument lets assume that the mantis shrimp can only distinguish a few different brightness levels on it's cones: off, low, medium, high and full brightness. In real life they can probably do much better than that, but if we stick to this assumption how many different colours could a mantis shrimp see?

Did you get 2441406252?

Yes, that's right! Twelve different detectors each of which have five levels gives us 5^{12} colours. That's getting on for a quarter of a billion colours even with such a coarse level of brightness detection.

I'm very tempted to call it astounding!

Did you get 2488332?

That's wrong I'm afraid. $12^5 = 248832$ but that's the sum you'd have to do for detecting 12 different levels. But we wanted 12 types of detector with 5 different levels. When you raise a number to a power you've got to get them the right way round!

Did you get 60?

Well 60 is the answer you get from multiplying 5 X 12 (or 12 X 5). But raising a number to a power is **very** different from multiplying them. And unlike multiplication it does matter which way round the numbers go.

Note i:　Some creatures really **do** only see in black and white (notably whales) because they only have one kind of light receptor

Note ii:　Ultraviolet is light so blue that it cannot be seen –a bit like most of Doug's jokes, Ha! Ha! Ha! Ha! Ha! No, seriously, it's very short wavelength light beyond the blue end of the spectrum that humans can see.

Here's a colour problem of a different kind. Katrina is giving her flat a makeover with a new colour scheme. The walls, ceiling, door and carpet are all going to be a new colour. All Katrina has to do is choose the combination of colours she wants.

This begs the question; "How many combinations of colours are there?". Well that partly depends on how many colours each individual item can be. Katrina's going to pick each of the four different colours from the so called "web safe" colours[see note i]. This is a limited set of colours that people often restrict themselves to when constructing their web sites (we do this too, on the Captain Astounding web site). They do this to be sure that the web site will look right on all the many different kinds of computer and different browsers that the web site will be seen on. Actually, the web safe colours are simply what you get by taking 20% increments from 0% to 100% on red, green and blue (computer screens work just like digital cameras or eyeballs except in reverse; they **emit** a different level of red, green and blue light for every pixel[see note ii]). This means that there are 6 different levels of each primary colour mixed together to give $6^3 = 216$ different options. This might sound like a lot, but of course it's a tiny fraction of the colours you can see (unless you're a whale) and web designers moan bitterly about it. So I'm not quite sure why a woman of taste like Katrina has limited herself like this —unless it's simply to provide me with a convenient example!

Katrina's fabulous flat makeover
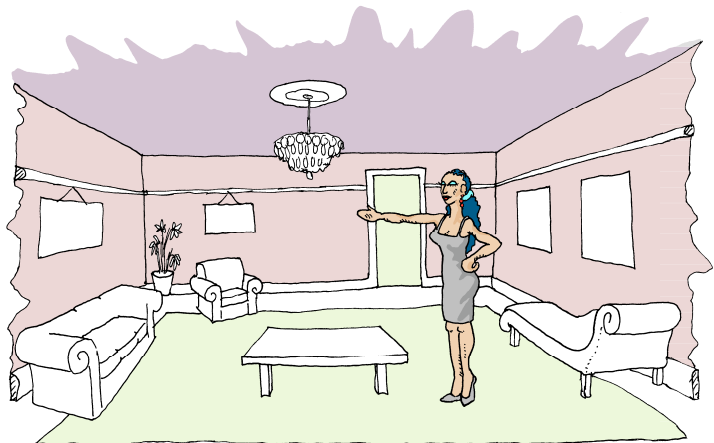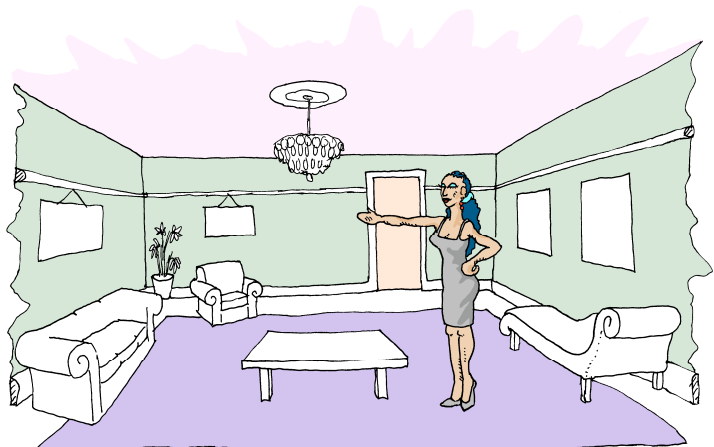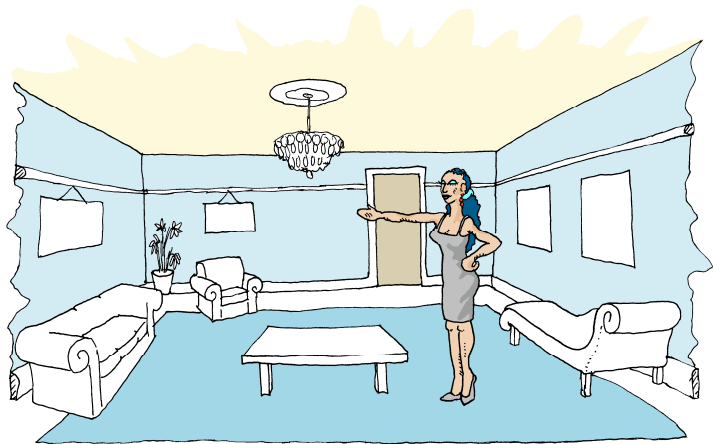
Note i: There's a beautifully clear discussion of web–safe colours and their uses in Robin Williams's delightful book "The non–designer's web book" (Reference 2 in the bibliography)

Note ii: I'm talking about computers for humans here; computer displays for dogs only emit a mixture of blue and yellow light. Computers for mantis shrimps are **extremely** expensive[see note iii]

Note iii: This is partly because they have to be waterproof of course.

So that's 216 different possibilities to choose from for each item. But we've got four colours to choose altogether: one for the ceiling, one for the walls, one for the door and one for the carpet. Suppose that we get raided by the really strict style police who insist on certain nice tasteful colours for the carpet, walls and ceiling. That would only leave us the door to play with, so there'd only be 216 choices. If on the other hand we get raided by the not–very–strict style police who let us choose our own carpet colour too then there will be 216 times as many possibilities. This is because for each possible door colour choice there are 216 colours we can have for the carpet. So there would be $216^2 = 46656$ different combinations to select from. (Note that there's nothing to stop us from using the same colour on the carpet that we choose for the door).

Well now you can probably see where this argument is going. If we get raided by the really apathetic not–very–strict–at–all style police who only control the ceiling colour, then there'll be $216^3 = 10077696$ combinations to choose from. And in real life, with no style police, we'll have an astounding $216^4 = 2176782336$ combinations:—

number of different items being given a new colour: walls, ceiling, door & carpet.

number of different colour receptor types in the human (Katrina's) eye

number of brightness levels

$$\left(6^3\right)^4 = 216^4 = 2176782336$$

So Katrina needn't be too worried about accidentally choosing the same colours as one of her neighbours! What happened here is that we raised a number to a power (ie 6 brightness levels raised to the 3 different cone types) and then raised that result to another power (namely 4, the number of different items: walls, ceiling, door & carpet). As you can see this business of doing a power to a power can quickly generate some enormous numbers!

Garish colours —a fun problem for people with no taste whatsoever!

Not everyone has Katrina's subtle and sophisticated tastes. If you'd rather have really loud primary colours then the way to do that is to use combinations of either max brightness or zero strength of red, green and blue. In other words you limit yourself to 0% and 100% levels and ignore the 20% increments in between. If at the same time we extend the scope of the flat decoration project by also giving new colours to the woodwork, the furniture, and the part of the wall above the picture rail, then how many combinations will there be? (Note that we're reupholstering all of the furniture in the same colour so the sofa, the chair and chaise longue are grouped together as one item).

Did you get 2097152?

That's the correct answer. We're looking for the number of colours raised to the number of different items getting a new colour. So you're absolutely right it's:—

$$\left(2^3\right)^7 = 2097152$$

Did you get 4782969?

Maybe you thought there were $3^2 = 9$ colours available so you got the answer $9^7 = 4782969$. But there are really only $2^3 = 8$ colours: black, yellow, cyan, blue, magenta, white, green and red. There are only 2 levels, 0% and 100% so there are $2 \times 2 \times 2 = 8$ mixtures of red, green and blue.

Did you get 4035360?

That's the answer you'd get by raising

$$\left(7\right)^2_3$$

the number of items (7) to the number of colours ($2^3 = 8$). It should have been the other way around! (see correct answer opposite).

Now this **definitely** sounds familiar!

Evening All!

Phil

Katrina's law

Katrina has noticed some thing a bit spooky about this process of raising a number to a power and then to another power. She's discovered that there's a different (and slightly easier) way of working out the number of colour combinations for her flat. If you take the original number of brightness levels (6 –for the web safe colours, remember?) and raise it to the power 12 (which is equal to 3 X 4, the number of cone types times the number of different items) then you still get the correct answer!

new method!

$$(6^3)^4 = 216^4 = 6^{(3 \times 4)} = 6^{12} = 2176782336$$

And when we were working out how many colours Noreen's camera could distinguish we noted that the number of brightness levels (128) was a power of two (= 2^7). So in actual fact we were doing a power to a power again. And once again we've got an alternative way of working out the right answer because we can raise two to the power 21 (which is equal to 7 X 3, the number of "bits" for each brightness signal times the number of different detector colours).

new method!

$$(2^7)^3 = 128^3 = 2^{(7 \times 3)} = 2^{21} = 2097152$$

If the number we start off with is 10 then it's a bit easier to see what's happening (because all we have to do is put the right number of zeroes on the end). For example if we take 10 cubed ($10^3 = 1000$) and then square it to get a million ($1000^2 = 1000000$) then that fits the pattern just fine:-

new method!

$$(10^3)^2 = 1000^2 = 10^{(3 \times 2)} = 10^6 = 1000000$$

So it really looks as if you can take a shortcut if you want to raise a number to a power and then again to another power; you just raise the number to the product[see note i] of those two powers. We call this Katrina's law (nobody else does, it's just us). But this all just a guess based on those few examples above. How do we know that it will **always** work, no matter what x and a and b are??

Katrina's law

$$(x^a)^b = x^{(a \times b)}$$

Funnily enough, it doesn't matter which way round the a and b go!

(Don't get either of them mixed up with x though!)

Do you remember Chef's law $x^a \times x^b = x^{(a+b)}$? It turns out that if Chef's law is true then Katrina's law must be true too. It works like this. First of all let's imagine a situation where we're doing a bit of $x^a \times x^b = x^{(a+b)}$ only a and b actually happen to be the same number. So it's really $x^a \times x^a = x^{(a+a)}$. But $x^a \times x^a$ is the same thing as saying x^a squared. And $x^{(a+a)}$ is the same thing as saying $x^{(2 \times a)}$. So we could write $(x^a)^2 = x^{(2 \times a)}$ which means that we can always work out the **square** of any number raised to a power just by **doubling** that power. But look, it works for cubes too! If $x^a \times x^a \times x^a = x^{(a+a+a)}$ then $(x^a)^3 = x^{(3 \times a)}$ which means that we can work out the **cube** of any number raised to a power by just **trebling** the power. And of course we don't have to stop with cubes. We could multiply x^a by itself **any** number of times (let's call that number b) and we'll always find that $(x^a)^b = x^{(a \times b)}$ which is what we wanted to prove!

Note i: **...a bit of terminology**

The **product** of any two numbers is just the result you get when you multiply them together. In other words "the product of a and b" means the same thing as "a times b".

The square root of all evil

When you know the answer and want to work out what the question must have been

Well here we are on the main dancefloor again. I've finally got round to painting that diamond of incredibly expensive iridescent paint that we talked about in chapter one. But there's a difference from chapter one: this time I'm going to try to paint the biggest possible diamond I can using one tin of paint. We never actually discussed how to do that before; we just thought up a size for the diamond (eg 4 metres), worked out what area it would cover (16 square metres) and compared it with the coverage printed on the tin (20 square metres) to see if there was enough (there was!).

In fact we then went on to try a 4½ metre diamond but it's area (= 4.5 squared = 20.25) was just a tiny bit too big for the tin. But at least we know that the size of the biggest possible diamond is somewhere in between 4 and 4½ metres. So in fact we could go on guessing like this and get closer and closer to the true maximum size x of the diamond that will have an area of exactly 20 square metres. For example if we make a new guess of 4.4 metres then this will need 4.4 squared = 19.36 square metres worth of paint. That's still a bit less than the tin contents so we could go on to try 4.45 metres. That would have an area of 4.45 squared = 19.8025 square metres so that's very nearly the right answer.

The answer is 20 square metres. Now, what was the question?

But I want to really get my money's worth out of the tin and paint **exactly** the right size diamond to use up all the paint[see note i]
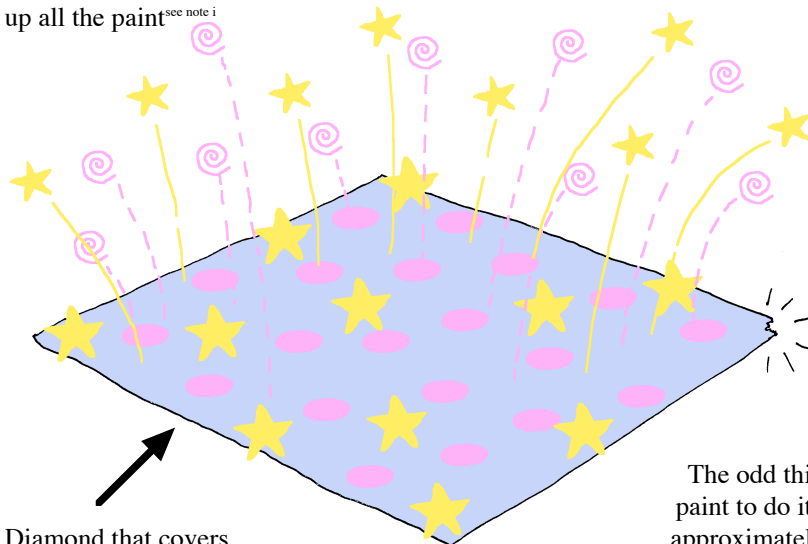
Diamond that covers exactly 20 square metres

The odd thing is; there's not quite enough paint to do it! The tin seems to be short by approximately the amount of paint it would take to cover a singer's fingernails.
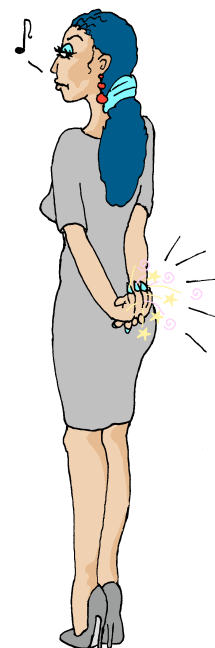
Note i: I'm saying nothing[see note ii]   *—Katrina*

Note ii: Oh all right then, I'll give you a little clue. It starts with "cheap", it ends in "skate" and rhymes with "cheapskate".

Some history and stuff

Admittedly, that weird square root symbol is a very compact way of writing it down. But it's a right pain in the neck if you want to use it in an email to somebody or type it in to a computer. For a start, there's no key on the keyboard with the radical[see note i] symbol on it. And there are hardly even any fonts that have a square root symbol. But we can't really blame that on old **Christoff Rudolff** who first introduced the symbol in his brilliant book "Die Coss" in 1525.

There is absolutely no evidence that Christoff Rudolff looked anything like this

Computers hadn't been invented and even printing was rather new and dodgy. You certainly didn't have many fonts to choose from (if you had a choice at all) because some poor beggar had to carve each character individually out of hard metal.

Actually, I really haven't any idea whether "Die Coss" is a brilliant book or not because I haven't read it. But Christoff Rudolff was one of the first blokes who realised that $x^0 = 1$, so he must have been a pretty good geezer.

It's as if we already know the answer we're looking for (20 square metres) but we've got to work out what the original question was. In other words we're doing things backwards. In chapter 1 the problem was something like:-

$$4^2 = 4 \times 4 = \text{What?}$$

And we quickly worked out that the answer was 16. But now we've got a problem that looks more like this:-

$$\text{What?}^2 = \text{What?} \times \text{What?} = 20$$

Which is to say "what is the number that gives us 20 when squared". Or in the more general problem (where we're not necessarily talking about 20); to get some value x what number is it that we need to square?

Introducing the fabulous SQUARE ROOT

The number you need to square (ie raise to the power 2) to **exactly** get a certain result x is called the **square root** of x. When you see "the square root of x" written down it'll often be using a weird symbol like this:— $\sqrt{x}$

For example we all know that 2^2 (= 2 X 2) is equal to 4. Therefore the square root of 4 is 2. So using the square root symbol we could write:— $\sqrt{4} = 2$

But of course the answer doesn't always turn out to be such a nice round number like two[see note i]. In fact our particular problem with the paint is a case in point because it's equivalent to finding the square root of twenty $(\sqrt{20})$ and there's no nice whole number that equals twenty when you square it. So how **do** we do it?

On your calculator...

Well luckily the good old calculator comes to the rescue again. Any half way decent one will have a square root button. It's nearly always labelled with the weird square root symbol[see note ii]. All we need to do is type in 20 and then press the square root button and we get exactly the right answer (as many digits as the calculator can display anyway). If you try this you should get an answer of about 4.472136 (depending on the number of digits) Try squaring it to check it's right!

Note i: On the few occasions when a number does have a square root which is a nice whole number then it's called a 'perfect square'. For example 4, 9 and 100 are all perfect squares.

Note ii: The square root symbol is sometimes referred to as 'radical'.

Here's a square root problem for you to try...

Problems involving a square root crop up just about everywhere. For example, take the case of my lovely new nightclub sign in chapter two. The force trying to blow the sign off went up with the **square** of the wind speed. So the force quadrupled when the wind speed doubled, and it increased by a factor of nine when the wind speed tripled. But what rise in wind speed would it take to just **double** the force on the sign?
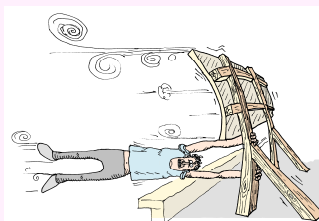
But we can use this back to front technique for other purposes, not just when there's a squared relationship. Let's go back to the Jungle Juice carton example. By the end of chapter one we'd settled on a carton with a five cm base. So the amount of juice it would hold was two times five cubed (= 250 cubic centimetres). But we might not have a specific size like 5 cm in mind. Instead, we might have some specific volume of juice to accommodate. Then we'd need to work backwards to find out the correct base width. For example, suppose we wanted a 1 pint[see note i] carton. Well 1 pint is 568.26 cubic centimetres. The shape of the carton was like two cubes stacked on top of each other remember (the part that contains juice anyway). So the width of base we're looking for is whatever number equals 284.13 when you raise it to the power 3. Now just like our square root problem with the p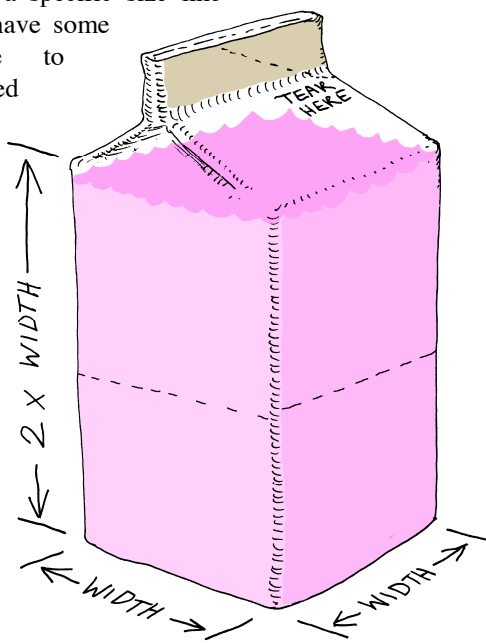aint, we **could** find out an answer by making successively better guesses. If we guessed 7cm for example then that'd give us a volume of $2 \times 7^3 = 686$ cubic centimetres. Well that's too big, so I could make second guess of 6cm. But that gives a volume of 2×6^3 which is 432 cubic centimetres and that's too little. So I could go on and try 6.5 centimetres and so on, but what I'm really after is the **exact** size of carton I need to just hold 1 pint, and I'd rather calculate it in one step if possible.

Did you get 1.41421?

...or something similar depending on how many digits your calculator displays? — It's the correct answer if you did. Doubling the speed means multiplying it by a factor of 2. So the wind speed only needs to rise by a factor of whatever number equals 2 when you square it. In other words $\sqrt{2}$

Did you get 4?

That's not the right answer. If the wind speed doubles then the force goes up by a factor of 4, but it doesn't work the other way around. We're looking for the wind speed that will double the force, and that means whatever number equals 2 when you square it

Did you get 2?

That's wrong. What you're saying is that doubling the wind speed will double the force on the sign. But wind doesn't work like that. The force rises much more steeply with rising speed, so in fact a much smaller increase in wind speed (see correct answer) will suffice to double the force.

Note i: That's one British pint I'm talking about. In the US they use a different pint that's quite a bit smaller.

Another 'power' problem

Remember the 'tale of two motorcycles' in chapter two? That was where we found out that to make your bike go faster by a certain factor x you need not x times the horsepower, not x^2 times as much, but x^3 times as much!
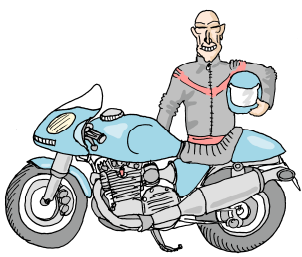
So how much faster can you go if you increase the **horsepower** by any factor x ? (answers at the bottom of the opposite page).

It's that venerable geezer **Christoff Rudolff** again!

In actual fact he didn't use the modern cube root symbol with it's little number 3. He wrote:-

$$\sqrt{x} \quad \text{and} \quad \sqrt{x}$$

for the cube root and fourth roots respectively. But unlike his square root symbol, they never caught on.

In the case of the 5cm carton all we had to do to work out the volume (well, half of it) was to calculate:–

$$5^3 = 5 \times 5 \times 5 = \text{What?}$$

Which is pretty easy. It's 125, so the total volume of the carton was 250 cubic centimetres. but to find out the correct size for the new 1 pint carton we've got to do something like:-

$$\text{What?}^3 = \text{What} \times \text{What} \times \text{What} = 284.13$$

Which is to say 'what is the number that gives us 284.13 when cubed'. Or in the more general problem (where we're not necessarily talking about 284.13); to get some specific value x what number is it that we need to raise to the power 3?

The even more fabulous CUBE ROOT

The number you need to cube (ie raise to the power 3) to **exactly** get a certain result x is called the **cube root** of x. The symbol used for a cube root is just like the 'radical' square root symbol except that it's got a little number three hovering over the squiggly front part like this: $\sqrt[3]{x}$

For example we all know that 2^3 (= 2 × 2 × 2) is equal to 8. This means that the cube root of 8 must be 2. Or to use the cube root symbol:– $\sqrt[3]{8} = 2$

But just like with square roots, not many whole numbers have nice whole cube roots so 8 is a bit of a special case. In fact the square and cube roots of a lot of whole numbers are **so** 'not whole' that you can go on adding extra digits of accuracy to the right of the decimal point but you'll **never** be able to write down the exact answer![see note i] If you don't believe me trying working out the square or cube root of 2 using the method of repeated guessing![see note ii]

On your calculator...

But happily that's not how we work out cube roots in real life because really good calculators have a cube root button like the one depicted above. All you have to do is type in the number you want the cube root of and press the button. Hey presto, you've got the exact right answer as accurately as your calculator can display it!

Note i: Numbers like this are called 'irrational numbers' but don't worry about it; that was the Sophisticates' lounge

Note ii: WHAT!! —you've given up ALREADY??.

So now we can solve the problem of the 1 pint jungle juice carton, as long as we've got a cube root button on our calculator. We just key in 284.13 and then hit the cube root key. That should come out at about 6.574, which means we'll need a carton with a 6.574cm wide base to hold exactly a pint. Problem solved! You can easily check to see if this is correct. Just raise the alleged answer to the power 3 (6.574 X 6.574 X 6.574) and then double it. You should end up with the number of cubic centimetres in a pint. That's 568.26.
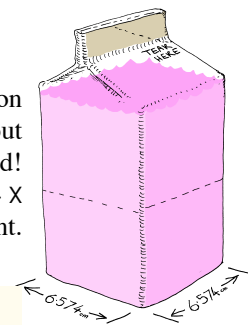
Terminology... Beyond the cube root.

Well if we can do a problem with x^3 backwards then why not try a problem involving x^4 or even x^5? With our dancefloor I beams for example, we found that the stiffness rose by a factor of x^4 when we blew up the cross section by a factor of x. So doubling the cross sectional size gave us a beam sixteen times stiffer. But that was a matter of choosing how much to increase the size (we chose 2) and seeing how much stiffer the beam got (it was sixteen times stiffer). But instead we might have said "Right, I want a beam that's three times stiffer, now how much bigger is it going to have to be?"). Well the answer is going to be whatever number x exactly equals 3 when you raise it to the power 4 (ie x X x X x X x = 3). It's going to be less than 2 because 2^4 = 2 X 2 X 2 X 2 = 16. But it's got be more than 1 because 1^4 =1 X 1 X 1 X 1 = 1. So the right answer is somewhere in between 1 and 2. But where **exactly** in between is it?
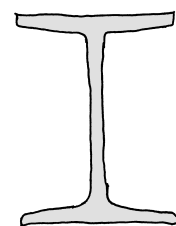
Well the official name for this number that needs to be raised to the power four to get x is the **fourth root** of x. And you may not be too surprised to know that it's written just like the square root symbol but with a wee number 4 hovering over the front part of the tick like this:— $\sqrt[4]{x}$

If you wanted to you could say 'third root' instead of cube root and 'second root' rather than square root. But people would probably look at you in a funny way. The expressions 'square root' and 'cube root' have been used for so long that it's probably too late to change them[see note i]. But going in the other direction there's no problem at all. The number that has to be raised to the power 5 to get x, for example, is called the fifth root of x. And of course it's written:— $\sqrt[5]{x}$
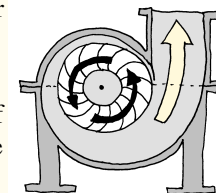
And the fifth root is exactly what you need if you want to know how much the size of a fan has to be increased to double it's energy consumption. You may remember from chapter 2 that the amount of electricity drawn by the motor of a fan goes up with the fifth power of it's size. So this will double when you scale up the fan by a factor of $\sqrt[5]{2}$

In fact you could have 23rd root if you wanted one! —there's no limit.

Almost certainly NOT on your calculator!

But there's a slight snag with these higher roots. You might have a cube root button on your calculator if you're lucky, but you're very unlikely to have a fourth root button let alone a 23rd root button. But luckily there's a truly astounding way round this! —join me in the Sophisticates' lounge overleaf and find out!

Did you get x?

Well it's not the right answer. If it were true then increasing the horsepower a certain amount would produce a proportional increase in speed. A 50% horsepower boost would give a 50% higher top speed for example. But in real life we get diminishing returns in top speed as the engine gets more powerful.

Did you get x^2?

That's the wrong way around. What you're saying is that if the horsepower increases by a certain amount x then the speed will shoot up by a factor of x cubed. But it's the horsepower that has to be boosted by x cubed just to get an x fold increase in speed. So each mph improvement in top speed costs you more and more the faster you go.

Did you get $\sqrt[3]{x}$?

Congratulations if you did, because it's the right answer. We were looking for the number which cubed is equal to x, and that's exactly what we mean by the "cube root" of x. It's whatever number that you've got to cube if you want to get an answer of x.

Note i: And if you hear somebody[see note ii] talking about 'root 2' or 'root 3' then that's a kind of slang for the square root of 2 or the square root of three, even though they didn't explicitly say second or square root. It's a bit dodgy but it's widely accepted.

Note ii: …probably an engineer.

y^x saves the day (again)

Well here's the trick you need to know if you want to calculate a root but haven't got the right button on your calculator. You're not going to believe that it works, but it does. You need to have a y^x button, mind you, but then a calculator without a y^x button belongs in the bin. The trick is this:—

$$\sqrt[n]{x} = x^{1/n}$$

There, I've said it! It might sound a bit weird but it really is astoundingly useful. It means that we don't even need a square root key really because we can just raise x to the power 0.5 (try it!). And it doesn't matter if the calculator doesn't have a cube root button because we can simply raise x to the power $\frac{1}{3}$.

So to work out the answer to that problem with the fan where we needed one which was scaled up by the fifth root of two… $\sqrt[5]{2}$

…then all we need to do is raise 2 to the power 0.2 (because $\frac{1}{5}$ = 0.2). Well if you try that on your calculator you should get round about 1.1487 depending on how many digits your calculator can display. In other words it only takes about a 15% increase in the size of a fan to double your electricity bill. Now **that's** Astounding!

But look! here comes Ceris with a financial problem that we can do backwards. If you've been trying out all the problems[see note i] you'll have had a bash at one where we worked out the growth on some money we held for 9 years in an account paying 17% interest. This equated to:—

$$(1.17)^9 = \text{What?}$$
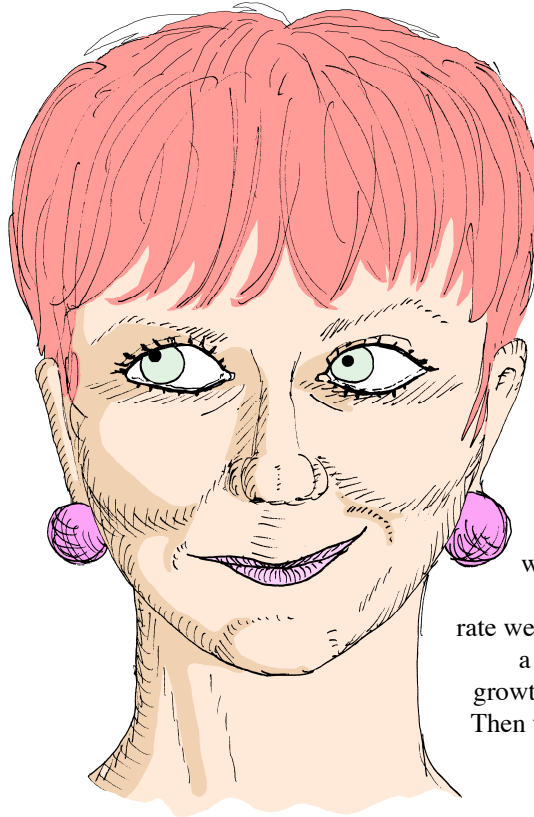
—which with the help of the lovely y^x button we quickly found out was a factor of 4.1084. And since we put in £500 at the beginning we ended up with £2054 and 20p.

But what if we weren't working with an exact known interest rate? What if we were trying to work out what interest rate we would need to get a certain amount of growth after nine years? Then we'd have to work backwards again

A Financial problem:…find the interest rate.

If you're investing some money and you want to be sure that you'll end up with at least three times as much after 20 years, then what's the minimum interest rate you can accept? —or to put it mathematically:—

$$\text{What?}^{20} = 3$$

Hint: You might like to read the green box! (left)

Did you get 5.65%?

Bang on! We need the number which gives you 3 when you raise it to the power 20. That's the same thing as saying we need the 20th root of 3. Since the calculator doesn't have a 20th root button, we just raise 3 to the power $\frac{1}{20}$. And $20^{0.05}$ equals 1.05647 so the answer is about 5.647%

He can't be banging on about that y^x button again can he? Please tell me he isn't!

I'm afraid he is!

Note i: —and you **have** been trying out all the problems. Haven't you??
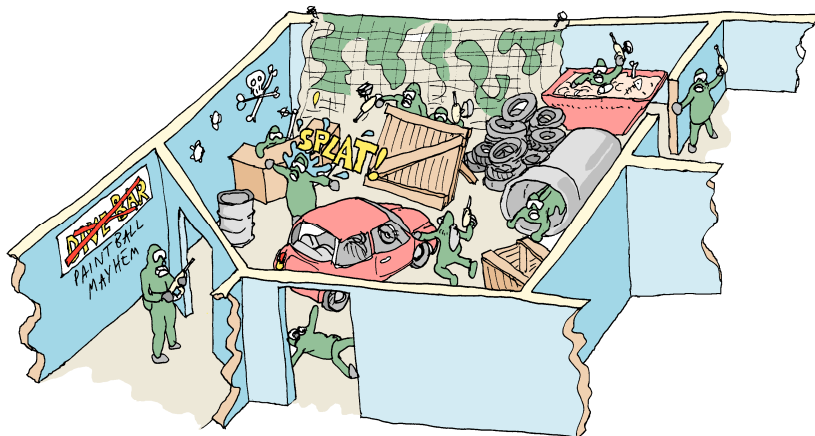
Strange powers

**...when the power isn't a whole number.**

What a way to start a party! Flamboyant Freddy is one of our regulars here at the club and he always likes to make a splash, or in this case a splat. So he's decided to kick off his birthday party with a paintball fight. That's why we've temporarily closed the dive bar and converted it into a specially designed paintball mayhem range. But that's not all. In a typically extravagant gesture, Flamboyant Freddy is having his own special paintballs made, filled with incredibly expensive iridescent paint. Oh yes.

Well if you've ever played paintball before you'll know that the guns are really just sophisticated pea–shooters that blast the paintball out of the barrel with a puff of compressed air (or sometimes CO_2). And the "ammo", the paintballs themselves, are spherical gel capsules filled with paint that burst on impact with the hapless target. Each player is equipped with plenty of compressed air, either in a reservoir inside the gun or in a separate cylinder on their back. And they each carry plenty of paintballs, which drop into the gun from a hopper fixed on top. It's great fun.

Now the normal size of paintball used has a diameter of 1.7 centimetres (or 17mm if you prefer). And when these little fellows impinge on the clothing of the victim[see note i] they create a rather impressive splat of about 10cm in diameter.

So oddly enough, this situation is very much like painting a diamond on your dancefloor but in miniature. The paintball is like a very small can of paint (only it's a sphere rather than a cylinder). And the splat is like a miniature dancefloor decoration (only it's round instead of diamond shaped).
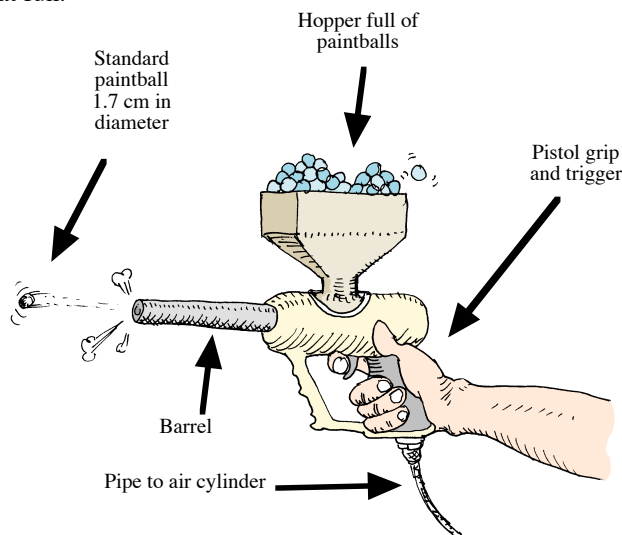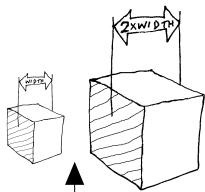

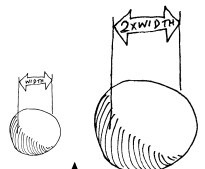**Note i:** …and it's only a tiny percentage of the paintballs that this happens to!:
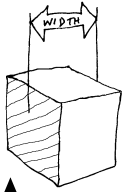
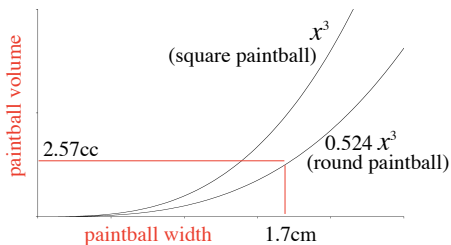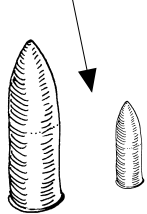Cube shaped paintball holding $1.7^3 = 4.913$cc of iridescent paint
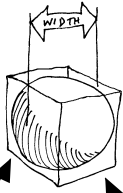
Cube holds more than sphere because of extra space at the corners

A double width cube holds eight times as much paint

In fact **any** shape of paintball will hold eight times as much if you double it's width

The volume of a Paintball

If the paintballs were cube shaped then it would be especially easy to work out how much paint they held. A 1.7cm wide one would hold 1.7^3 = 4.913 cubic centimetres of the expensive stuff. But luckily[see note i] paintballs are balls and not cubes. This means that they don't hold as much as 4.913 cubic centimetres. In fact if you think about a 1.7cm wide ball then it's obviously not going to hold as much paint as a cube of the same width because the cube has got quite a lot of extra space available near its corners.

It turns out that a ball always has about 52% of the volume of a cube of the same width, no matter what width we are talking about. So our standard 17mm paintball only holds about 2.5 cubic centimetres of paint. Actually, 52% is just an approximation. If you're a stickler for the **exact** answer then you'll need to step into the sophisticates' lounge (see below right) and meet one of the most famous and useful numbers of all.

But no matter what shape your paintballs are one thing that remains true is that doubling the width will give you eight times the volume. And trebling the width will give you 27 times the volume. Or widening them by any factor x will give you x^3 times the volume.
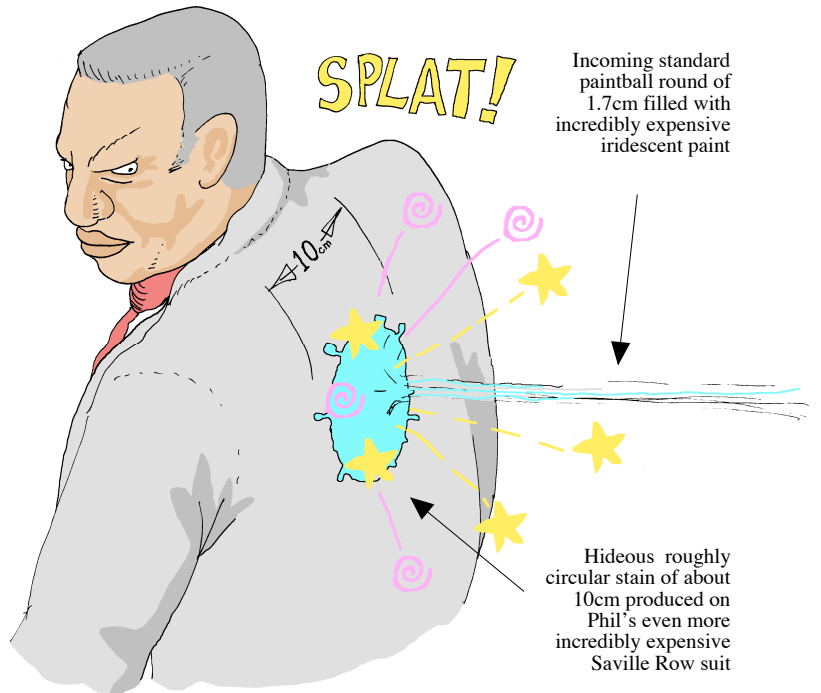

Note i: If paintballs were cubic then they'd be a rotten fit in the barrel of the gun and most of the compressed air would leak out round the side of them. You could have a square paintball gun barrel I suppose but I still don't think it's a good idea. The paintballs would probably jam in the chute from the hopper.

Note ii: In fact a bloke **did** write a book about π, and very good it is too (see Petr Beckmann's "A history of π" —reference 3 in the bibliography, page 67)

But whenever weapons are involved you're sure to get an arms race developing pretty soon. And sure enough, having resisted the temptation to cheat for several seconds, Doug and Chef have given in and resorted to the big guns (see opposite page). Doug is brandishing a 2.7cm bore paintball shotgun in a most unsavoury manner, while Chef is staggering under the weight of a preposterously large paintball bazooka firing a tennis–ball like round of 5.7cm!

The question is this: what difference does this make to the size of the splat produced on impact?

Incoming standard paintball round of 1.7cm filled with incredibly expensive iridescent paint

Hideous roughly circular stain of about 10cm produced on Phil's even more incredibly expensive Saville Row suit

A brush with π,   the superstar of numbers

We know the exact amount of paint in a square paint "ball"; it's x^3 ccs. The exact amount in the case of a sphere is $\frac{1}{6}\pi x^3$. The figure π (pronounced "pie") simply stands for a number which is roughly 3 and a bit. To be more exact it's about 3.14. Or to be even more exact it's about 3.14159. But you could go on and on being more exact and adding more digits but never quite reach the true value of π. That's because it's one of those spooky "irrational" numbers. π crops up whenever you're comparing the shape and size of squarish things with rounded ones. It crops up in a lot of other places too, some of which are very surprising. You could write a whole book about it[see note ii]


Not **that** kind of pie
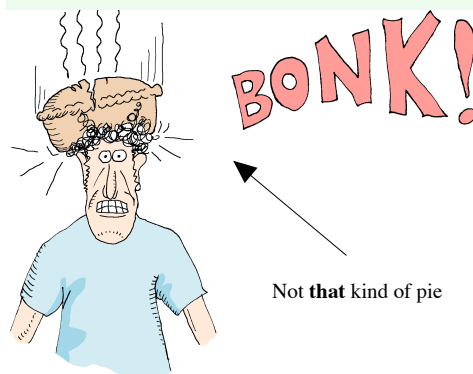
A quick exercise about paintballs.

Here's a problem you can have a go at if you like. How much iridescent paint does one of Doug's paintballs contain? (Doug's the one with the 2.7cm bore gun, remember).

…answers overleaf.

The area covered by a paintball splat

Well now let's think about the area covered by the splat. We know that a standard paintball gives you a splat 10cm in diameter. If the splat was square it would be very easy to work out its area; it would just be $10 \times 10 = 10^2 = 100$ square centimetres. But of course the splat is round not square. So it actually only covers a smaller area. A circle only covers about 78½% as much area as a square of the same width. This is because a square has extra areas near the corners that are absent in a circle. So our splat only covers about 78.5 square centimetres. In reality that 78½% is only an approximation though. The **exact** area of a circle is $\frac{1}{4}\pi^2$ times the diameter (see sophisticates' lounge box on the previous page) so in fact you can work out the answer as accurately as you like!

PH-LAPP!
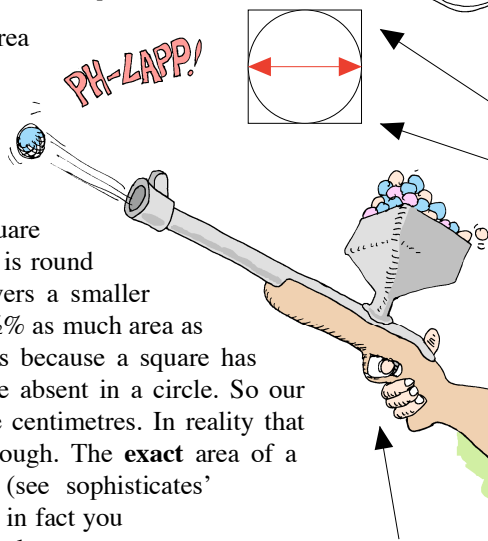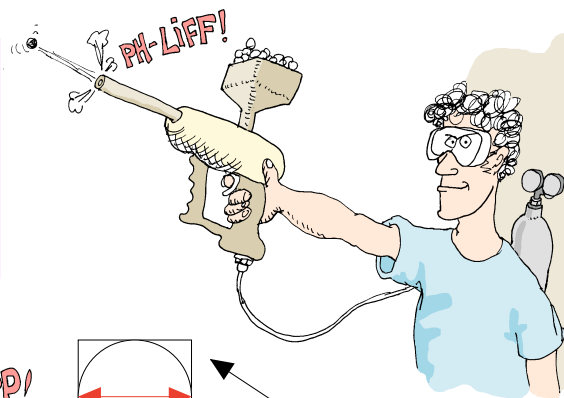
A square of a certain width has a greater area than a circle of the same width because of these extra bits near the corners

Doug's rather vulgar 2.7cm bore paintball shotgun

What's the coverage?

Well now we know what the area of a standard splat is. And we also know what the volume of a standard paintball is (it's about 2.57 cubic centimetres). So now we can work out the "coverage" of the iridescent paint. This is a measure of how much area you can cover with a certain volume of paint, which is a property of how "runny" it is. Since the area was 78.5 square centimetres, our coverage rate is $78.5 \div 2.57 = 30.54$ square centimetres per cc of paint.

PH-TOoo!
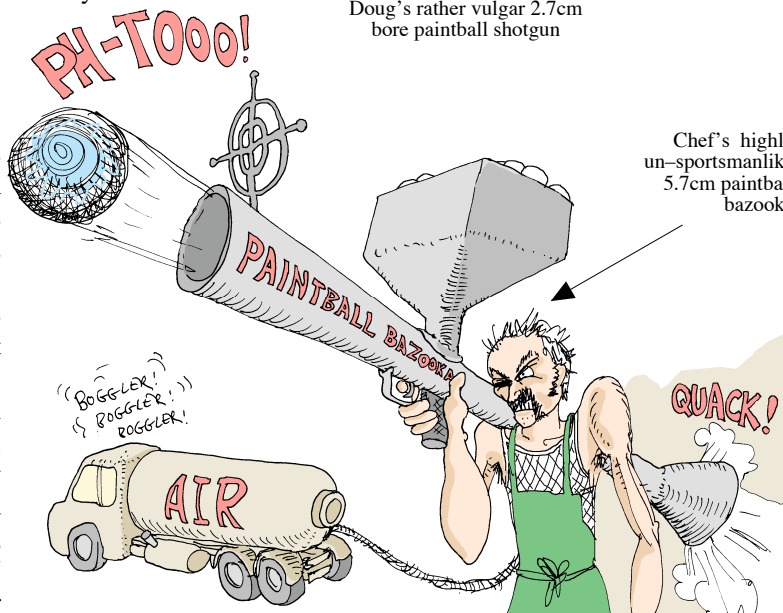
Chef's highly un–sportsmanlike 5.7cm paintball bazooka

PAINTBALL BAZOOKA

BOGGLER! BOGGLER! BOGGLER!

AIR

QUACK!

Assumption

We're assuming that when a paintball hits you the paint spreads out nice and evenly almost as if you'd put it on with a brush[see note i]. If it caused a thick blob in the middle and got thinner towards the edges   then our assumption would be wrong. But examination of Phil's suit (after he'd calmed down) showed that the splats are indeed pretty even.

Upside down

If paint volume goes up with the square of splat width, then splat width must go up with the square root of paint volume. You can see this in the two graphs below with their *x* and y axes swapped
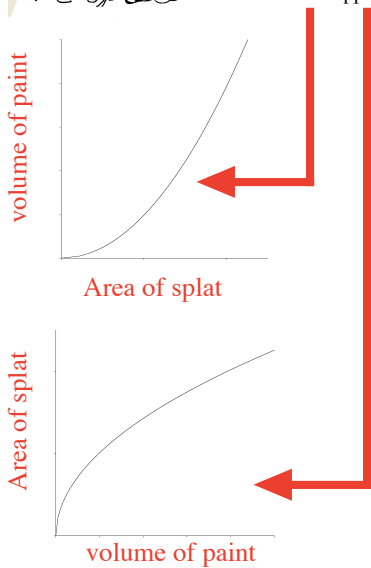

But no matter what coverage we get from the paint it'll still always be true that you'll have to quadruple the volume of paint if you want to double the size of splat. That's because the area of the splat rises with the square of the splat's width.

$$\text{Paint volume} \quad \overset{\text{goes}}{\underset{\text{with}}{\text{up}}} \quad \text{Splat width}^2$$

But of course we're really looking at this problem upside down. We're asking what volume of paint it would take to produce a splat of a certain width. But we would really like to know how wide a splat we can make with a certain volume of paint. So if the paint volume goes up with the square of splat width then that must mean that the splat width goes up with the square root of paint volume. And since taking the square root of a number is just the same thing as raising it to the power 1⁄2 (see chapter 6) we get:–

$$\text{Splat width} \quad \overset{\text{goes}}{\underset{\text{with}}{\text{up}}} \quad \text{Paint volume}^{0.5}$$

Well this chatter about volumes of paint and splat areas is all very interesting, but it isn't what I originally set out to discover. What I wanted to understand was the relationship between paintball width (= gun bore) and the diameter of splat it produces.
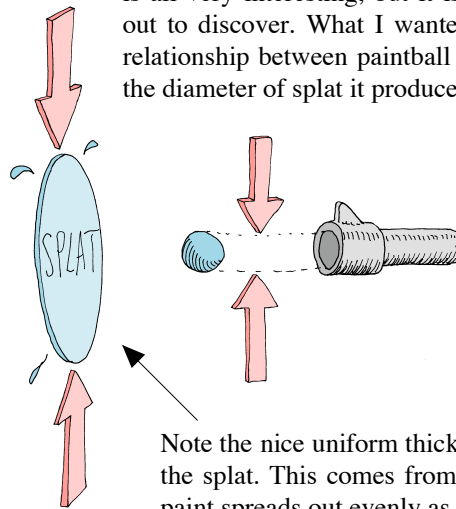
Obviously, the bigger the paintball the bigger the splat will be. But if we plotted a graph of ball size versus splat size what sort of curve would we get?

Note the nice uniform thickness of the paint layer in the splat. This comes from our assumption that the paint spreads out evenly as if it had been brushed on.

Did you get about 3.82?

That answer is too small. One way of arriving at that answer is to multiply the bore (2.7cm) **squared** by 1⁄6π. But the volume is calculated by multiplying the diameter **cubed** by 1⁄6π. A small error in the power to which you raise a number can sometimes cause a big error in the answer!

Did you get about 19.683?

That's too much. What probably happened is that you cubed the paintball width all right and got your answer of 19.683. But then you forgot to multiply it by the factor of 1⁄6π necessary to give you the correct answer for a sphere rather than a cube.

Did you get about 10.3cc?

…or something similar depending on how many digits your calculator displays? — It's the correct answer if you did. Cubing the paintball diameter gives us $2.7^3 = 19.683$. Then we just need to multiply this by 1⁄6π (which is about 0.5236) to get the correct volume. Well done.

Note i:: …which would be a pretty daft way of playing paintball!

Well we've just worked out how the splat width varies as we increase the paint volume: it's:-

$$\text{Splat width} \quad \text{\textcolor{red}{goes up with}} \quad \text{Paint volume}^{0.5}$$

But at the beginning of this chapter we also discussed how the volume of paint increases as we raise the diameter of the paintball: it goes up with the **cube** of the bore:-

$$\text{Paint volume} \quad \text{\textcolor{red}{goes up with}} \quad \text{Paintball width}^3$$

So we can just fill in this formula for paint volume into the first expression and we get:-

$$\text{Splat width} \quad \text{\textcolor{red}{goes up with}} \quad \left(\text{Paintball width}^3\right)^{0.5}$$

In other words the splat width goes up with the square root of the cube of paintball width.

But let's not forget Katrina's law (from chapter 6). That was where we saved the effort of raising a number to a power **a** and then raising it again to another power **b**. We just raised the number to the power (**a** ✗ **b**) instead.

Katrina's law: $\left(x^a\right)^b = x^{(a \times b)}$

…the rather astounding conclusion of this is that splat width goes up with paintball width to the power **one and a half**! This means that if we double the paintball size then splat will get $2^{1.5}$ times bigger. That's a factor[see note i] of about 2.83

$$\text{Splat width} \quad \text{\textcolor{red}{goes up with}} \quad \text{Paintball width}^{1.5}$$

Well a graph of x to the power 1.5 will curve upwards a bit like a graph of x squared, only not quite as steeply. It won't be like the graph of the square root of x (ie x to the power 0.5) where the graph droops off as you get diminishing returns from pushing x higher and higher.

But if you try to explain what $x^{1.5}$ means in terms of how many times you multiply x by itself then you might get a bit stuck. It's easy to understand that $x^1 = x$, $x^2 = x ✗ x$, $x^3 = x ✗ x ✗ x$ and so on. But where does that leave $x^{1.5}$? Similarly, with the kitchen telegraph we found that two sticks gave us 4^2 possibilities, three sticks gave us 4^3 possibilities and so on. It wouldn't really make sense to have a one and a half stick telegraph though, would it?. If you think about this stuff for too long you may find that your brains pop out (see diagram A).
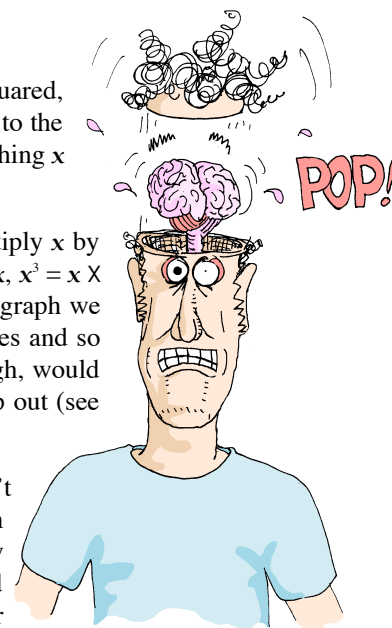
But in real life, nobody worries too much about these strange powers that aren't whole numbers. They're so useful that everyone just gets on and uses them happily without excessive brain–popping. In the previous chapter we saw how useful powers like 0.5 and 0.25 can be when you're calculating square roots and higher roots of numbers (even if you haven't got the right button on your calculator). The graph of x^2 curves upwards and the graph of x^3 curves upwards more sharply. So everyone accepts that the curve of $x^{2.5}$ curves upwards somewhere in between the two.

Diagram A

Note i: You can try this out on your calculator using the lovely y^x button. Mind you, this begs the question: "how does the calculator know how to do it?" Now that's an interesting question. It's bit beyond the scope of this book really, but there's a brief explanation in appendix II (the super sophisticates' lounge).

In fact there are examples of strange powers all over the place. Here I am in the air conditioning system next to the big chiller that cools down the air before it gets pumped around the club. Well it turns out that the cooling effect of this gadget increases as $x^{0.8}$ for a given air speed x.
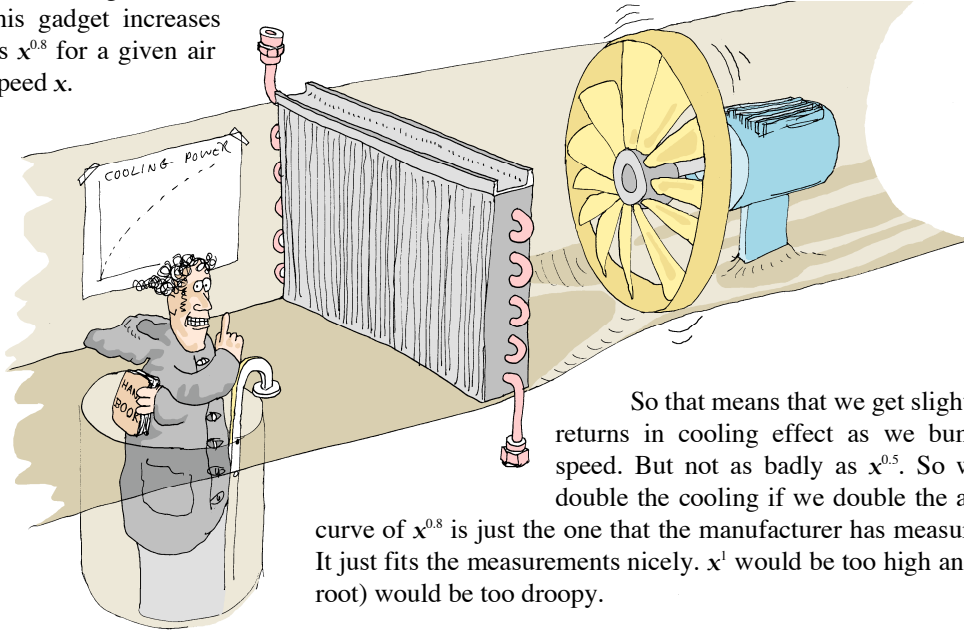
So that means that we get slightly diminishing returns in cooling effect as we bump up that air speed. But not as badly as $x^{0.5}$. So we won't quite double the cooling if we double the air speed. This curve of $x^{0.8}$ is just the one that the manufacturer has measured in his tests. It just fits the measurements nicely. x^1 would be too high and $x^{0.5}$ (ie square root) would be too droopy.

What power do you need?

Logarithms

Well here's Ceris again doing her usual clever stuff with the money. And just like before she's looking at the interest rates on all the accounts to see if we can get a better deal, whether it's a higher rate for our savings (when we've got spare money) or a lower rate for loans (when I've spent too much money).
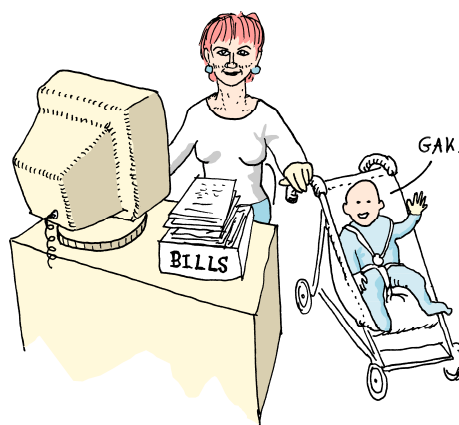
And right now Ceris is looking into "Dodgy Dave's Doubler" It's a high interest scheme that gets it's name from the fact that you double your money each year. So at the end of the year you've got 200% of the amount you had at the beginning. In other words it's a 100% interest rate, because you keep the money you started with[see note i] and get another 100% on top. Well this is pretty good, in fact Ceris says it sounds too good to be true. Of course if you leave your money in for several years then it gets much better still because of that thing about the graph going up in a curve due to the miracle of compound interest. And having read chapter 3 we can easily work out how much the money grows over a certain number of years. It's a matter of calculating—

$$2^{\text{Number of years}} = \text{What?}$$

For example if we kept our money with Dodgy Dave for four years we'd end up with $2^4 = 16$ times as much money (assuming Dodgy Dave hasn't left the country). But then in chapter 6 we saw how you can do this sort of calculation backwards so that you can tell what interest rate would be needed to get a certain amount of growth. Suppose we started getting really greedy and wanted the money to grow not by a factor of 16 but by a factor of 20 over those four years. That would require an even higher interest rate. It would be a matter of calculating:—

$$\text{What?}^4 = 20$$

which (if we were getting a bit sophisticated) we'd say out loud as 'what is the fourth root of 20?' $\sqrt[4]{20}$ And, (if we were getting even more sophisticated) we might even be able to work out the answer. Undeterred by the absence of a fourth root button on our calculator, we could just use the trusty y^x button to raise 20 to the power of a quarter ($20^{0.25}$). This would give us the same answer, which is about 2.115. This means we need an interest rate that gives us 2.115 times as much money at the end of every year. So that means we'd need an interest rate of 111.5% (because you don't count the money you start with) to get our twenty times growth in just four years.

But there's one kind of calculation we haven't tackled yet. How could we calculate how many **years** it would take to get a certain amount of growth given a certain interest rate? (In the other problems the number of years has been a fixed part of the question, not the answer)

Note i: …plenty of schemes don't even guarantee **this**, so take care!:

Logarithms

There's a name for this situation where you're trying to find out what power you need to raise a number to, and that name is **logarithm**.

The number which gets raised to that power is called the base, so in the example opposite the answer we're looking for is the **logarithm to the base 2** of 64. Which turns out (neatly) to be 6.

Actually, the word **logarithm** is such a mouthful that it's nearly always called **log** for short. And the base being used is often added as subscript like this: $\log_2 64 = 6$.

Not **this** kind of log!
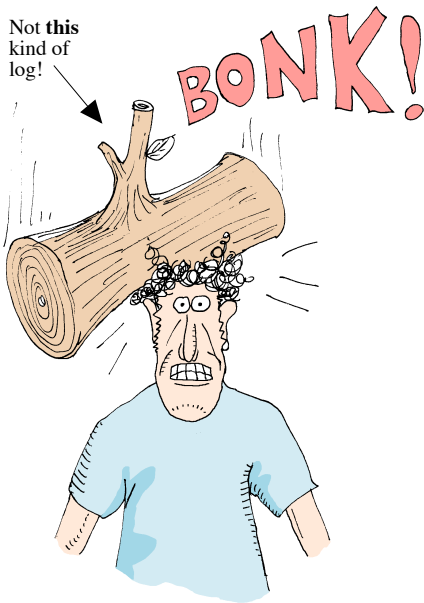
You can have a log to any base you like. For example; we know that $5^3 = 125$, therefore the log to the base 5 of 125 ($\log_5 125$) is equal to 3.

For example, we might want to know how many years it would take for our money to increase 64–fold using Dodgy Dave's Doubler. The interest rate is fixed (100%, ie doubling each year) and the growth is fixed (at our fantasy factor of 64) so the only variable is the number of years. In other words:-

$$\text{What?} \atop 2 \quad = \quad 64$$

Well I suppose this particular problem wouldn't be too hard to crack by trial and error[see note i]. I mean, we already know that the what? is going to be greater than 4 because we worked out that $2^4 = 16$ earlier, and that's not enough. So we could just guess some answer that's a bit higher like 7 (in other words we're guessing that it'll take 7 years). But if you work out 2^7 it comes out at 128, which is too much. The right answer has therefore got to be somewhere in between 4 and 7 seven years. And sure enough, if we get lucky and pick 6 as our next guess then it just fits perfectly. $2^6 = 64$ so it'll take precisely 6 years to multiply your money 64–fold in Dodgy Dave's Doubler.

On your calculator (well, ...possibly)

Of course, there's got to be a better method than this trial and error approach. If you're really lucky you might even have a button on your calculator which will do it with a single press. If you've got one, the "log to the base 2" button (see terminology discussion, left) will look something like this:

Not very many calculators have it but a few do.

You may discover that you've got a button with just plain "log" on it and no indication of what the base is. If so this is probably a "log to the base 10" function. They should really print $\log_{10}$ on it but it's sometimes left off as a kind of shorthand. But be careful because occasionally you find a "log" button that does logs to the base **e** (which is a very important irrational number round about two and three–quarters[see note ii]) instead of logs to the base 10. They should really print $\log_e$ on it in this case, but it doesn't always turn out that way. On the other hand, if you see a button with **ln** printed on it then that will definitely give you a log to the base e (2.71828....) and not a log to the base 10.

It's very important to know what base you want your logs in! If you do the wrong sort of log then you'll get a very misleading answer! But don't worry if you haven't got the right button —there's a crafty trick coming up shortly!
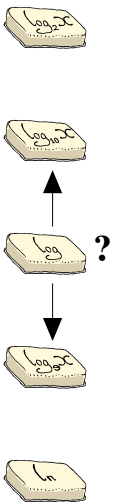
Note i: Especially if you recognised that 64 is a power of 2

Note ii: The number e is an extremely important one that crops up all over the place. At first you might think it's only useful for seeing how many years you need to leave money in an account with 171.828% interest. But it turns out to be useful in many ways (a bit like π is). It's beyond the scope of this book really, but you could fill an entire book just talking about e. In fact a bloke **did** write a book about it, and very good it is too:- (see Eli Maor's "e: the story of a number", reference 4 in the bibliography, page 67)

Here's a quick exercise…

Suppose you've got a calculator that has a "log" button on it but you're not sure what base of log you're going to get when you press it. You want logs to the base 10 but you're worried in case you get logs to the base e by mistake. So as a test you type in 1000 and then press the log button. If it does what you want ($\log_{10}$) then what answer would you expect to see?

Summary: 3 kinds of problem…

Looking for…:	a POWER	a ROOT	a LOG
Written as:	$x^y = ?$	$?^x = y$	$x^? = y$
Alternative notation…	or… $x\hat{\ }y = ?$ $x**y = ?$ (see note i)	$\sqrt[x]{y} = ?$	$\log_x y = ?$
In words…	x to the power y equals what?	What to the power x equals y? or… The xth root of y equals what?	x to the power what equals y? or… log to the base x of y equals what?
An example. (to do with interest rates)	What growth am I going to get on my money at 9% interest for 6 years? $x = 1.09$see note ii y = 6 answer = $1.09^6 = 1.6771$ So I end up with 67.71% extra money.	What interest rate would I need to **double** my money in six years? $x = 6$ y = 2 answer = $\sqrt[6]{2} = 2^{0.166666666…}$ = 1.1225 = 12.25%	How long would it take to get 500 times as much money at 900% interest? $x = 10$see note ii y = 500 answer = $\log_{10} 500$ = 2.699 yearssee note iii

Did you get 3?

That's the right answer because 10 raised to the power three (…cubed in other words) is equal to 1000. That's exactly the same thing as saying log to base 10 of 1000 is 3.

Did you get 10?

That's not the right answer. If the button does what we want then we should see whatever power you have to raise 10 to in order to get 1000. That number should be 3, because you've got to **cube** 10 to get 1000. The answer you gave is the cube **root** of 1000, but that's not what we were hoping for…

Did you get 6.9078?

Well if you got this number then your "log" button is doing logarithms to the base e not to the base 10. You can check this using your y^x button by raising e (2.71828…) to the power 6.9078 which will give you 1000 or something very close. So it's the wrong answer, because we hope to see $\log_{10}$ 1000 (equals 3, see right answer)

Note i: These two forms are common where you get to type in a formula at a keyboard (like the "calculator" applications on computers or in computer languages.) They get around the problem of how to type in the little superscript figures for the powers. They're also handy if you need to text somebody an equation on your mobile phone!

Note ii: Interest rates are invariably quoted as the amount you get in addition to your original money. So if your money grows by a factor of 10 then we call that 900%, not 1000%. And a 9% interest rate means you've got 1.09 times as much money at the end of the year

Note iii: Of course, you probably only get your interest paid once a year. So the bad news is that you'd really have to wait 3 years before you could really get your hands on the money. The good news is that it would have increased by more than 500 times by then.

Any base you want

Luckily, there's a truly astounding trick for working out a logarithm to any base you want. As long as you've got a log button of some kind on your calculator then you can use this trick; it doesn't actually matter what kind of logarithm the button gives you. The trick works because of this remarkable fact:-

$$\log_a b = \frac{\log_c b}{\log_c a}$$

…where a and b and c can be any number. It works like this: Let's say you've got a $\log_{10}$ button on your calculator. All you need to do is replace the two "c"s in the above formula with 10. Now let's assume that for some reason you want to work out the logarithm to the base 5 of 17 (in other words to what power does 5 have to be raised to get 17). All you would need to do is work out the log to the base 10 of 17 (which is about 1.23045) and then divide that number by the $\log_{10}$ of 5 (which is about 0.69897). If you work that out you should end up with something pretty close to 1.76038 And that's the answer.

But is it correct?

You can easily check this result. Just use your lovely y^x button to raise 5 to the power 1.76038 and you should get 17 or something very close.

Oi! Captain! This logarithm malarkey is all very interesting but I'm not going to be able to make use of it very often, am I? I mean; even if I was lucky enough to have buttons for three different kinds of log ($\log_2$, $\log_{10}$ and that weird $\log_{2.7128\ldots}$) then that only helps me if I happen to have interest rates of 100%, 900% or 171.8281828% respectively. And quite frankly, none of those rates sounds very likely, except maybe from the local loan shark.

Fear not dear reader! …as long as your calculator gives you at least **one** kind of log then you can actually work out logs to any base you like. That'll let you compute how long you need to leave your money for realistic interest rates like 5%, 11.5% or whatever is available at the time you're reading this. But If you want to learn how to do this you **will** need to pay a quick visit to the sophisticates' lounge, I admit. (see left)

The crafty thing about this is that it makes no difference what the number "c" is. We end up dividing two different numbers but the answer is always the same. So if you've got an anonymous "log" button then you don't even need to check whether it gives you a $\log_{10}$ or a $\log_e$. You'll get the correct answer either way!

So now we can (finally) work out the number of years for a plausible down–to–earth interest problem. For example, if you want to double your money on 3.5% interest rate then you need to work out the log to the base 1.035 of 2. The answer will be the number of years you've got to wait. You can use your log button to calculate the logarithm of 2 (to some base, which we don't care) and then we divide that by the logarithm of 1.035 (calculated in exactly the same way). If you actually follow that through you should be looking at an answer not far off 20.15. So the rather depressing result is that it'll take 21 years (assuming you're paid interest yearly) before your stash is worth twice as much. And Ceris is muttering darkly about something called "inflation". I'm not sure what she's talking about but apparently inflation makes it even worse.

It's not just for interest rate problems!

Here's a problem for you to try using logs and it's got nothing to do with money or interest rates. Thinking back to the great kitchen telegraph (4–sided stick version) how many sticks would you need if you had **ten thousand** items on your menu?   (answers at bottom of facing page)

He can't be droning on about that y^x button **yet again** can he? Please tell me it isn't true!

I'm afraid he is. **again**!

Logarithms instead of a calculator   (a bit of quite recent history)

We're so used to having calculators and computers lying around that it's hard to imagine doing calculations without them. But it wasn't that long ago that people had to do their calculations by hand. The really tricky ones cropped up when you had to have very accurate calculations because of the long string of digits that you had to deal with. Sometimes great accuracy isn't really necessary. For example, if you're working out the length of a piece of wood to repair your shed then an answer within about 1% will probably do. But in other situations getting a really accurate answer is crucial. In navigation for example you might need an answer that's right to within a tiny fraction of a percent to make sure you don't end up on the rocks.

When you're doing calculations by hand it also makes a lot of difference what kind of calculation you're doing. Generally speaking, addition is quite easy but multiplication can get very hard. For example, let's take two familiar numbers (e & π) and make sure we write them down nice and accurately (let's say 4 decimal places). Well if I ask you to add them together using only a pencil and paper you'd probably do it very quickly. But if I asked you to multiply them I'd probably have time to nip up to the sophisticate's lounge, have a glass of jungle juice, or maybe two, and you'd still be checking your working when I got back. If you try this [note i] your working will probably look something like this:–

Addition (not too hard)   **Multiplication** (pain in the neck)
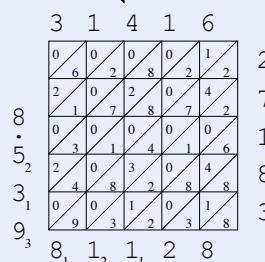
And that was only four decimal places, which is a lot less than your calculator can handle. The worst part about it is that the amount of work you need to do goes up roughly with the square of the accuracy you need. With addition on the other hand you only double the amount of work when you double the precision.

Luckily however, there's a neat trick based on chef's law whereby you can transform the multiplication into an addition. In effect you draw a big graph of x versus 10^x It's a bit like the graphs of sticks/strings versus number menu items in chapter 4, but with the whole curve drawn in between.

Did you get 4?

4 is the answer you'll get if you take log10 of 10000. But that's the number you need to raise 10 to for answer of 10000. What we want is the number you have to raise **4** to for answer of 10000. You can calculate this as log10 10000 divided by log10 4 (see right answer).

Did you get 10?

Well that's what you'd get by taking the fourth root of 10000 (which is the same thing as 10000 raised to the power 0.25). But that's not what we needed. We were looking for the number which gives 10000 when you raise 4 to that power.

Did you get 7?

That's the right answer. A telegraph made up of n four-sided sticks can cope with 4ⁿ dishes. So we're asking what power does 4 have to be raised to for an answer of 10000, in other words "what's log of 10000?". Well log 10000 is about 6.644, but since you can't have 0.644 of a stick you'll need 7 whole sticks to do it.

Note i: Try to get out a bit more

...a graph of x vs 10^x
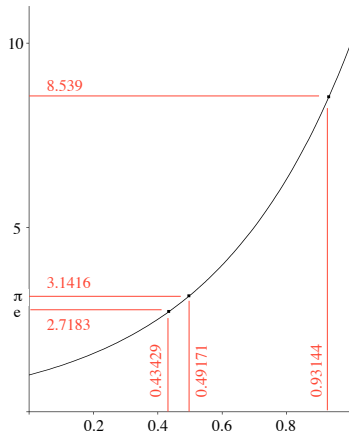
...or is it log$_{10}$x vs x?

This graph shows how you can multiply two numbers by adding their logs. We find out the logs (to the base 10) of 2.7183 and 3.1416 on the y axis by looking up on the curve how far along the x axis you have to go to reach that height. The answers are 0.43429 and 0.49715 respectively. We add these logs together to get 0.93144. This is the log$_{10}$ of the answer. So we just read off the height of the curve at 0.93144 on the x axis; it's about 8.54

Not **that** kind of table
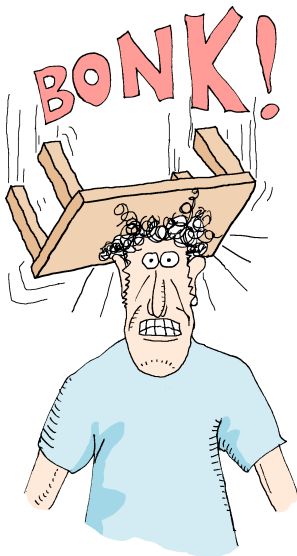
The idea is this: You find the number that you need to raise 10 to for an answer of 3.14159. That's the same thing as asking what's the log to the base 10 of π. It's about 0.49715. And we also find out what power 10 has to be raised to if we want to end up with 2.71828. And that's no different from working out the log to the base 10 of **e**, which is about 0.43429.

Chef's law: $x^a \times x^b = x^{a+b}$

But Chef's law[see note i] tells us that adding the powers together is equivalent to multiplying the actual numbers. This means that if we add those two logs together (0.49171 + 0.43429) then we'll end up with the number you've got to raise 10 to if you want an answer of π × e. In other words the log10 of (3.14159 × 2.71828) is equal to 0.49171 + 0.43429 = 0.93144. So all we have to do to get the answer is to look up what number has a log10 of 0.93144. That's the same thing as working out $10^{0.93144}$ giving us an answer of about about 8.539

So we didn't end up doing any manual multiplication at all. We just looked up the logs of the numbers and added them together. Then we just had to look up what number had that sum as it's log. And that's how we managed to avoid doing all the tedious work of a long multiplication.

Well of course you wouldn't be able to get a very accurate reading of log$_{10}$ from a graph like the one above left (unless it was really huge —and then it'd be a nuisance to carry about). So what happened before calculators was that people carried little booklets of **log tables** around with them. Log tables were in effect just a long list with ordinary numbers down one side and the corresponding log to the base 10 down the other. So you simply looked up the logs of the two numbers you wanted to multiply, added them together, and then looked up what ordinary number corresponded to that sum.

Of course, there was slightly more to it than I've described and there were a lot of other clever things you could do with log tables, but Chef's law was the basic principle by which they worked.

Note i: Ok, I admit it's really only me who calls it "Chef's law".
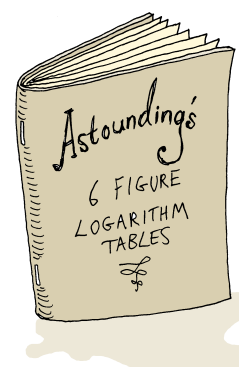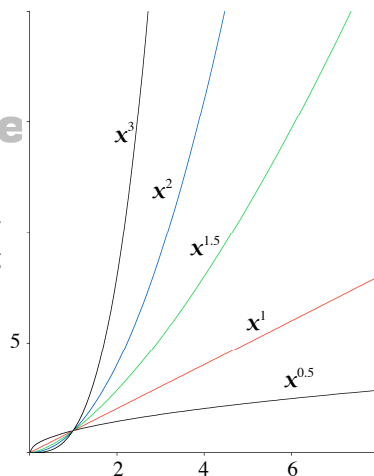
# Appendix I

Summary and useful formulae

This book has all been about raising a positive number to a positive power. It's not difficult to define what this means as long as the power is a whole number like x^2 or x^3. We can simply define it as multiplying the number by itself a certain number of times. Or, slightly more conveniently, we can think of x^y as multiplying 1 by x y times over:-

$x^0 = 1$ $x \times 0 = 0$
$x^1 = 1 \times x$ $x \times 1 = 0 + x$
$x^2 = 1 \times x \times x$ $x \times 2 = 0 + x + x$
$x^3 = 1 \times x \times x \times x$ etc $x \times 3 = 0 + x + x + x$ etc

This "repeated multiplication" definition shows up an interesting comparison between the process of raising to a power and the process of multiplication. Multiplying a number is like adding it to itself repeatedly whereas raising a number to a power is like multiplying it by itself repeatedly. So the relationship between addition and multiplication is very much like the relationship between multiplication and exponentiation[see note i]. This definition isn't so helpful when it comes to non–integer powers like 0.5 or 3.14 though, but it's clear enough when you see them on a graph. The green curve in the graph above is x to the power 1.5. You can see that is falls neatly in between x^1 and x^2. The graph also shows how powers higher than 1 curve upwards as x increases whereas those smaller than 1 droop off with diminishing returns. Note too that this means that when x is smaller than 1 (see the bottom left hand corner of the graph) the higher powers give a smaller value than the lower ones. For instance 0.5 squared is 0.25 whereas 0.5 cubed is only 0.125
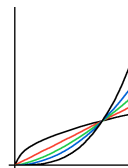
1: Chef's law: $x^a \times x^b = x^{a+b}$

Chef's law is certainly a useful trick to know about. But it's really much more than that. In a very real sense Chef's law **is** the definition of what raising a number to a power means. In other words the answer to the question "what does it mean to raise a number to a power" should really be "it's whatever process that makes Chef's law true". And that definition works just fine for strange non–integer powers too.

2: $\dfrac{x^a}{x^b} = x^{a-b}$

Chef's law means that you can multiply the numbers by adding the powers. And it follows that you can also divide the numbers by subtracting the powers (see equation 2, left). After all, if putting three extra strings on a kitchen telegraph multiplies your options by a factor of 8 ($=2^3$) then taking three strings off again must divide the number of choices by 2^3 to give you the original capacity

3: Katrina's law: $\left(x^a\right)^b = x^{(a \times b)}$

Katrina's law (as we call it) is a statement of the fact that if you take a number and raise it to a power and then raise the result to another power then you'll get the same answer as if you'd raised the original number to the **product** of those two powers. For example, we can easily work out 10000 squared because 10000 is just 10^4 and "squaring" is just raising to the power 2. So the result is $10^{(4 \times 2)} = 10^8$ (or a hundred million in other words)

Note i: Exponentiation is just another word for the process of raising a number to a power.

4: Working out an nth root

$$\sqrt[n]{x} = x^{1/n}$$

In chapter 6 we looked at the square root and other roots of a number. In other words we were trying to solve problems of the form what?^{y} = x rather than x^{y} = what?

Now in the case of square roots we can probably simply resort to the square root button on the calculator and if we're lucky there might even be a cube root button too. But for all other roots it's well worth remembering that you can calculate roots as fractional powers as in equation 4, left.

5:   Logs: change of base

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Chapter 8 was all about logarithms. That's where we don't know what power we've got to raise something to to get a certain answer. In other words we were solving problems of the form $x^{what?} = y$ or to use the mathematical lingo; "what is the logarithm to the base x of y" ($\log_x y$)?

And just as with roots the calculator gives us a little bit of help but doesn't quite do the whole job. Your average calculator will do you logs to base 10 or maybe logs to base e or possibly both. But most don't do logs to any other bases. That's why the logarithm "change of base" formula (5, left) is a very useful one to know.

Appendix II

...is that all there is?

Even the sophisticates don't know all of this stuff!
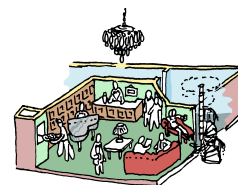
Welcome to the "super–sophisticates' lounge". This is the part of the book where we confess about the things that we failed to mention in the text, even in the sophisticates' lounge. And the answer to the question "is that all there is?" is an unhesitant YES, because there's a lot more to learn about powers beyond the topics covered by this volume.

Powers of a negative number

All of the applications of exponentiation we've explored in this book have involved raising a positive number to a power. But you can raise a negative number to a power if you want to (for example -2 squared equals 4) in fact it goes on all the time. You can see what happens in the graph opposite which we've stretched leftwards to include negative values of x. The interesting thing is that even powers like x squared and x^4 go **up** as x gets more and more negative (in fact they're symmetrical about the y axis) but the odd powers like x^1 and x cubed will go **down**. So when you've got a negative number you know that the result of raising it to an odd power will also be negative, but the raising a number to an even power will always give you a positive result.
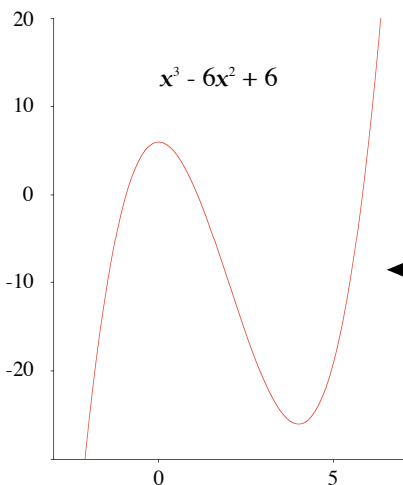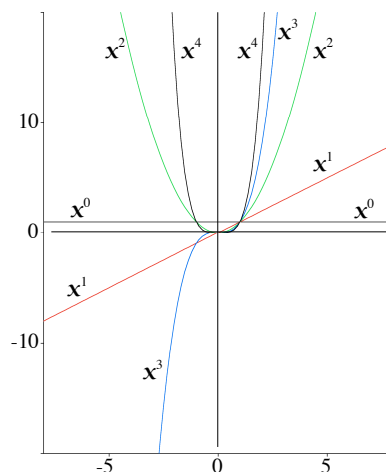
Polynomials

This behaviour of odd and even powers is the key to the fantastic flexibility of polynomials (see sophisticates' lounge, page 21). Polynomials are equations where you add or subtract various different powers together and you can use them to create curves with almost any shape you like. For example the graph opposite shows a plot of $x^3 - 6x^2 + 6$.

$x^3 - 6x^2 + 6$

Negative powers

Not only did we avoid running in to powers of negative numbers, we also skirted around the issue of **negative powers** of numbers. Negative powers, like non–integer powers can be a bit tricky to think about. After all, it's hard to imagine a kitchen telegraph with minus 3 sticks. But by convention we define a negative power x^{-n} as $1 / x^n$. For example, 2^{-3} equals 1/8. Actually, this is not just a convention; it's the only definition that will make Chef's and Katrina's laws work. So it must be right! Sometimes you will see negative powers in the units a quantity is expressed in. For example, this book is printed on paper that weighs 160 grams per square metre. That's sometimes printed as $160gm^{-2}$, because the weight is divided by the number of square metres.

$x^0 = 1$
$x^{-1} = 1 / x$
$x^{-2} = 1 / x^2 = 1 / (x \times x)$
$x^{-3} = 1 / x^3 = 1 / (x \times x \times x)$
$x^{-4} = 1 / x^4 = 1 / (x \times x \times x \times x)$

Negative powers

x^{-1}

x^{-2}

x^{-3}
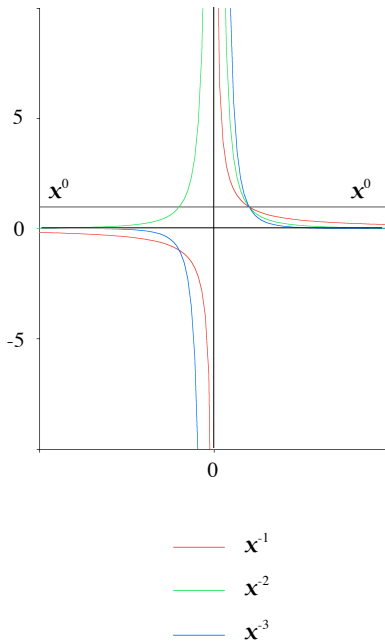
You can see the curves made by some negative powers on the graph opposite. Interestingly, they still have the property that the even powers always stay above the x axis but the odd powers have negative values when x is negative, just like positive powers.

Roots of negative numbers

Taking powers of negative numbers gets even more tricky when the power in question is not a whole number. Take square roots for instance (ie raising a number to the power 0.5). Let's say we're looking for the square root of -3 for example. We can try to get the answer by guessing a value like 1.7, but that's no good because 1.7 X 1.7 gives us a positive number, 2.89. But if we guess a negative number like -1.8 then that's no good either because -1.8 X -1.8 still gives us a positive number, 3.24. And we can see that the graph of x^2 (previous page) is entirely above the x-axis (being an even power) so there's no point on the curve that has negative height, y. But astoundingly enough, this doesn't mean that the square root of -3 doesn't exist, it merely means that it's not on the graph. Numbers like this are called complex numbers and they're well beyond the scope of this book[see note i], but they actually turn out to be extremely useful

How does your calculator do it?

Throughout the book we've encouraged you to pick up your calculator and tap in things like $x^{0.25}$ (for a fourth root) or $x^{0.8}$ for the air conditioning example in chapter 7. But we sneakily neglected to mention how the calculator works out the answer. It's easy to see how an integer power can be calculated because it's just a matter of multiplying x by itself the right number of times, but non–integer powers need a different approach. Happily, it turns out that you can work out a non–integer power as accurately as you wish by summing up a series of ordinary integer powers. In fact your calculator is probably doing something loosely equivalent to this:-

$$ y^x \;=\; 1 \;+\; \frac{x \log_e y}{1} \;+\; \frac{(x \log_e y)^2}{(1 \times 2)} \;+\; \frac{(x \log_e y)^3}{(1 \times 2 \times 3)} \;+\; \cdots $$

Adding up a series of integer powers like this is called a power series and it's an extremely important and powerful technique. Of course, the one above is no use unless you know how to calculate $\log_e x$ (which is another thing we've glossed over by saying "do it on the calculator". But don't worry! —you can work out $\log_e x$ quite simply with another power series:-

$$ \log_e x \;=\; x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \frac{x^5}{5} \;-\; \cdots $$

Note i: You can read all about this fascinating subject in Paul Nahin's excellent book "An imaginary tale: the story of $\sqrt{-1}$ "

(see reference 5 in the bibliography, page 67).

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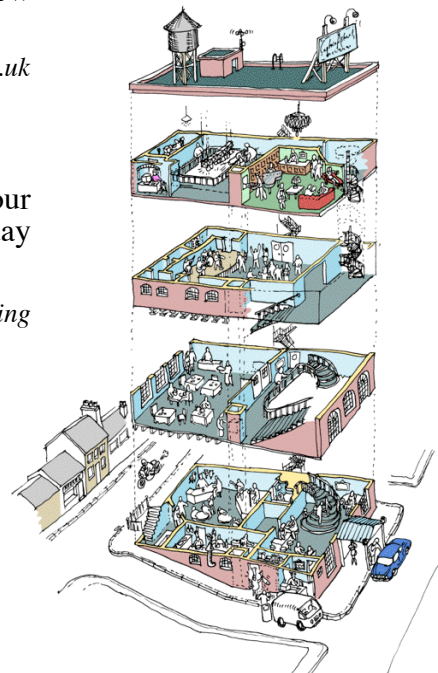
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